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Inflationary Cosmological Perturbations in Modified Theories of Gravity

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Abstract

This thesis is devoted to study the inflationary applications of cosmological perturbation theory within the framework of modified gravitational theories. Both metric and Palatini variational principles are indroduced and employed in different theories of gravity including $f(R)$ theories, Scalar-Tensor theories and Generalized Gravity. For all these theories the modified action is presented, equations of motion are derived and some main features are discussed. A short review on standard inflation and cosmological perturbation theory is provided. Inflationary cosmological perturbations are considered in all the gravitational theories mentioned above whose scalar and tensor spectral indices and tensor-to-scalar ratios are found. All the results are summarized, compared and discussed in the conclusion.

Preface

This thesis represents the result of the work the author undertook at Imperial College London under the supervision of Carlo Contaldi. Although some preliminaries has yet been studied from the beginning of 2010, full time effort to the project has been dedicated from June to the end of November 2010. The scope of the whole thesis is the submission for the italian *Laurea Specialistica* at University of Trento in December 2010.

The original part of the work has converged in a research article which is, at the moment of writing, still in pubblication. Anyway, the eprint is already available online at the reference

> N. Tamanini and C. R. Contaldi, *Inflationary Perturbations in Palatini Generalised Gravity*, arXiv:1010.0689 [gr-qc],

and will be denoted with [TC10] throughout the text. As it can easily guessed from the title of the paper, the main studies of that work focus on the applications of cosmological perturbation theory to the primordial inflationary phase within the framework of Palatini Generalized Gravity, which includes the more famous $f(R)$ and Scalar-Tensor theories.

The present thesis is instead devoted not only to study the argument in the Palatini approach, for which all the original analysis carried out in [TC10] will be reviewed, but also to introuce the already derived results within the well-known metric formulation. The target is to face with both the variational principles and to eventually compare the obtained results, in order to outline the differences between the two formulations in different modified theories of gravitation.

The technical level steadly improves from the beginning to the end of the work. Initially the style seems to coincide with a book af popular science, so no formulae at all are given in Chapter 1. Then, specifically in Chapter 2 but also in Chapters 3 and 4, the approach becomes harder but still pedagogical in order to introduce and guide the reader to the more technical arguments of Chapters 5 and 6, where several details are kept implied and more references to other works are provided.

The outline of the thesis is as follows. Chapter 1 contains a completely non-technical introduction to some major issues of modern cosmology. After a brief history of cosmology, the problems recently arised observing the universe, namely Dark Matter, Dark Energy and Inflation, are discussed. Then the main motivations, both theoretical and experimental, for modifying the General Theory of Relativity are presented in order to familiarize with the forthcoming topics. In Chapter 2 both the metric and Palatini formulations of General Relativity are studied showing how, regardless of the differences, they eventually lead to the same physical theory.

In Chapter 3 a wide range of modified theories of gravity is offered. These run from Scalar-Tensor theories to Generalized Gravity passing through Brans-Dicke and $f(R)$ theories. For all of them the modified action is introduced and equations of motion are derived. An important equivalence between $f(R)$ and Brans-Dicke theories is also studied since of crucial importance for the physical results of the last chapters. Chapter 4 constitutes a short review of standard inflation and perturbation theory. First the Flatness and Horizon problems are defined showing how a primordial expansion of the universe can solve both of them. Then the standard inflationary model is analysed and the basic elements of cosmological perturbation theory are set up. Finally, the perturbations are applied to the inflationary epoch deriving the canonical results for the scalar and tensor spectral indices as well as for other cosmological observables.

The last two chapters represent the principal part of this thesis. In Chapter 5 the theory of cosmological perturbations is developed within the framework of metric Generalized Gravity and the main results for inflationary cosmology, namely the scalar and tensor spectral indices, are found. Then some examples are provided reducing the analysis into $f(R)$ and Scalar-Tensor gravity and the particular model of Starobinsky inflation is studied. Chapter 6 proceeds in parallel to Chapter 5 treating cosmological perturbations within the Palatini formulation of Generalized Gravity. Again, scalar and tensor spectral indices are computed and the examples of $f(R)$, Scalar-Tensor and Non-minimal inflations are discussed.

Finally, a short chapter of conclusion is given where all the work in summarized and the main results are compared. An interesting table showing the scalar and tensor spectral indices for any theory introduced is also constructed in order to render the physical differences between these theories of immediate impact.

The bibliography, though sufficiently wide, cannot be thinked as a self-consistent guide to the literature. Its only aim is to provide useful references for the reader interested in deepening the material treated throughout the thesis, but it can also be used as a starting basis for further readings. The citations are denoted with the initials of the authors and the year of publication, whilst the bibliography is organized in alphabetic order.

Aknowledgments

There are some people I would like to thank for the different contributions they gave to this work.

First and foremost, I am truly greatful to my Imperial College tutor Carlo Contaldi. I owe him a lot of things starting from all the (few) Friday beers he bought me[1]. He treated me more like a friend than a student and from the professional relationship we shared I learnt how real research is effectively made. Thanks also for the correction of my "italian" English in the first chapter of this work. I really want to express my gratitude for all the support he gave me during the year I spent at Imperial College and for the one I am sure he will not spare in the future.

I would like to thank also my italian supervisors Guido Cognola and Sergio Zerbini who have carefully read the manuscript and given me useful advices. I also cite Valter Moretti for some interesting discussions we had on the mathematical foundations of the variational principle in General Relativity. Then, although they have no directly contributed to this work, special thanks to Mattia Dalla Brida and Andrea Dapor for the innumerable talks about physics and other (less interesting) stuff.

I appreciate all the people and the istitutions which made my work possible. The Imperial College Theoretical Physics Group at Blackett Laboratory created the perfect environment for making research and I will never forget its people, starting from the kindly secretary to the friendly PhD and master students. Thanks also to the Erasmus Programme which, though with a little economical contribution, gave me the opportunity of living London. I think every student, regardless of its subject of study, should take a long term experience abroad to face another culture.

Last, but definitely not least, I am really grateful to my family. My sister and my parents have ever supported me, not least economically, during these years of studies. I am in their debt for all the love they have given me since I was born and I cannot figure out a better family than this. Finally, I want to express my infinite love to my mate Valeria for all the experiences we shared and for all the ones still waiting us. The patience and the effort she showed in helping me to write the non-technical part of this thesis is unforgettable.

To her and my family this work is dedicated.

[1]Which is clearly the most important.

Notation

An attempt has been made to keep the basic notation as standard as possible. However, the use of non-metric connections did require use of some non-standard notation. The following list will hopefully be a useful tool for clarifying this non-standard notation. In general, the notation, standard or not, is always defined at its first occurrence in the text and in all places that ambiguities may arise, irrespectively of whether it has been included in this guide. The signature of the metric is assumed to be $(-,+,+,+)$. We set $c = 8\pi G = 1$, with c the speed of light and G the Newton's gravitational constant, throughout all this thesis unless otherwise specified. Greek indices α, β, λ run from 0 to 3 while latin indices i, j, k run from 1 to 3 (spatial components).

$g_{\mu\nu}$		Lorentian metric
g		Determinant of $g_{\mu\nu}$
$(\mu\nu)$		Symmetrization over μ, ν
$[\mu\nu]$		Antisymmetrization over μ, ν
$\Gamma^{\lambda}_{\mu\nu}$	$:= \frac{1}{2} g^{\lambda\sigma}(2\partial_{(\nu} g_{\mu)\sigma} - \partial_\sigma g_{\mu\nu})$	Levi-Civita Connection
∇_μ		Covariant derivative w.r.t. $\Gamma^{\lambda}_{\mu\nu}$
${R_{\mu\nu\alpha}}^{\beta}$	$:= 2\partial_{[\mu}\Gamma^{\beta}_{\nu]\alpha} + 2\Gamma^{\beta}_{[\mu\lambda}\Gamma^{\lambda}_{\nu]\alpha}$	Riemann tensor of $\Gamma^{\lambda}_{\mu\nu}$
$R_{\mu\nu}$	$:= {R_{\mu\lambda\nu}}^{\lambda}$	Ricci tensor of $\Gamma^{\lambda}_{\mu\nu}$
R	$:= g^{\mu\nu} R_{\mu\nu}$	Curvature scalar of $\Gamma^{\lambda}_{\mu\nu}$
$G_{\mu\nu}$	$:= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$	Einstein tensor
Ψ		Matter fields (collectively)
$T_{\mu\nu}$		Energy-momentum tensor
$\hat{\Gamma}^{\lambda}_{\mu\nu}$		Independent connection
$\hat{\nabla}_\mu$		Covariant derivative w.r.t. $\hat{\Gamma}^{\lambda}_{\mu\nu}$
$\hat{R}^{\beta}_{\mu\nu\alpha}$	$:= 2\partial_{[\mu}\hat{\Gamma}^{\beta}_{\nu]\alpha} + 2\hat{\Gamma}^{\beta}_{[\mu\lambda}\hat{\Gamma}^{\lambda}_{\nu]\alpha}$	Riemann tensor of $\hat{\Gamma}^{\lambda}_{\mu\nu}$
$\hat{R}_{\mu\nu}$	$:= \hat{R}^{\lambda}_{\mu\lambda\nu}$	Ricci tensor of $\hat{\Gamma}^{\lambda}_{\mu\nu}$
$\hat{R}$	$:= g^{\mu\nu}\hat{R}_{\mu\nu}$	Curvature scalar of $\hat{\Gamma}^{\lambda}_{\mu\nu}$
ϕ		General scalar field
$\omega(\phi), V(\phi)$		General functions of ϕ
$T^{(\phi)}_{\mu\nu}$		Scalar field energy-momentum tensor

$f(R,\phi)$		General function of R and ϕ
$F(R,\phi)$	$:= \frac{\partial f(R,\phi)}{\partial R}$	
$a(t)$		Cosmological scale factor
K	$= 0, +1, -1$	Spatial curvature
$H(t)$	$:= \frac{\dot{a}}{a}$	Hubble parameter/rate
η	$:= \int \frac{dt}{a(t)}$	Conformal time
P		Perfect fluid pressure
ρ		Perfect fluid energy density
N		Number of e-foldings
P	$:= \frac{1}{2}\omega\dot{\phi}^2 - V$	Scalar field pressure
ρ	$:= \frac{1}{2}\omega\dot{\phi}^2 + V$	Scalar field energy density
ϵ_1	$:= -\frac{\dot{H}}{H^2}$	Slow-roll parameter
ϵ_2	$:= \frac{\ddot{\phi}}{H\dot{\phi}}$	Slow-roll parameter
ϵ_3	$:= \frac{\dot{F}}{2HF}$	Slow-roll parameter
ϵ_4	$:= \frac{\dot{E}}{2HE}$	Slow-roll parameter
$\alpha, \beta, \gamma, \psi$		Scalar perturbations
b_i, c_i		Vector perturbations
h_{ij}		Tensor perturbations
$\Phi_{(\alpha)}$	$:= \alpha - \frac{d}{dt}\left[a(\dot{\gamma} + \beta)\right]$	Scalar gauge invariant
$\Phi_{(\psi)}$	$:= -\psi + aH(\dot{\gamma} + \beta)$	Scalar gauge invariant
$\Phi_i^{(V)}$	$:= b_i - a\dot{c}_i$	Vector gauge invariant
$\mathcal{R}$	$:= \psi + \frac{H}{\rho+P}\delta q$	Curvature perturbation invariant
$\mathcal{P}_s$	$:= \frac{k^3}{2\pi^2}\lvert\mathcal{R}\rvert^2$	Scalar power spectrum
$\mathcal{P}_t$	$:= 8 \times \frac{k^3}{2\pi^2}\lvert h\rvert^2$	Tensor power spectrum
$n_s - 1$	$:= \left.\frac{d\ln\mathcal{P}_s}{d\ln k}\right\rvert_{k=aH}$	Scalar spectral index
n_t	$:= \left.\frac{d\ln\mathcal{P}_t}{d\ln k}\right\rvert_{k=aH}$	Tensor spectral index
r	$:= \frac{\mathcal{P}_t}{\mathcal{P}_s}$	Tensor-to-scalar ratio

Contents

Chapter 1

Introduction

1.1 300 Years of Gravitation

Since Newton's famous apple, we have known that the distant stars we see in a sparkling starry night are connected, to each other and with us, by the same phenomenon which forces our feet to stay stuck on the ground. This interaction, capable of crossing such long distances, has been historically known with the name of gravity. Gravity dominates the physical behaviour of our universe from distances involving a few centimeters to billions of light years. It forces the rain to fall, the Tower of Pisa to tilt, the Earth to orbit around the Sun and the galaxies to form. It is the strangest and most elusive force of nature. No wonder it has taken hundreds of years to realize how it works in depth. To this day, as we will see, a complete understanding has far from been achieved.

The Newtonian theory of gravity was probably the first unification of two different phenomenological interactions: "local" and celestial gravity. It was conceived to describe how physical objects behave under the influence of gravity. Remarkably, it worked for more than two hundred years: all the astronomical observations of the solar system collected at the time were successfully explained by the theory. There were no signs to suspect that Sir Isaac Newton's theory was not the definitive interpretation of gravity. The only unexplained fact, the motion of the perihelion of Mercury, could be justified postulating the existence of a new missing inner planet[1]. No one, at the end of the nineteenth century, could guess that in order to explain this anomaly the whole Newtonian theory had to be deeply revolutionized.

The change came with the new century. In 1916, whilst the great war was raging throughout Europe, Albert Einstein gave birth to the final form of his General Theory of Relativity (GR). This brand new theory was conceptually different from every classical theory formulated until then (and

[1] It is interesting to note how this missing planet, called Vulcan, was perhaps the first example of Dark Matter (see next section).

from the nascent Quantum Mechanics too). The most revolutionary features introduced were the Equivalence Principle, which identifies inertial and gravitational mass, and the new gravitational field equations connecting spacetime geometry to matter distribution. These were able not only to provide the right orbit of Mercury, but also to predict a slight deviation in the motion of light rays grazing the sun. This new behaviour of light was observationally confirmed by the famous expedition of Sir Arthur Eddington carried out in occasion of the 1919 solar eclipse. Thanks to this, it was clear that Einstein's theory had to be considered the new theory of gravitation.

Since then, GR has been used to compute all the necessary calculations involving large distances and high energies. Remarkably, as it happened with Newton's theory, it has survived almost a century satisfying all the tests conceived to discredit it. At the moment it is indeed well known that GR, at least within intergalactic distances, is the best theory describing gravity. Furthermore, animated by its successes, physicists and astronomers started to formulate mathematical models of the universe as a whole. The only tools they had were Einstein's theory of gravity and the assumed large scale homogeneity and isotropy of the universe (Cosmological Principle). Their aim was to unveil all the cosmic secrets. These men founded a new science: Cosmology.

We can divide these first cosmological models into two classes: the dynamical models, in which the universe as a whole evolves in time, and the statical models, in which the universe lied and will eternally lie as it stands. In the second class we find Einstein's attempt to freeze the cosmic evolution introducing the nowadays so fashionable (see next section) cosmological constant. However, after Hubble's 1929 discovery of cosmic expansion, all the static models were abandoned[2] and became clear that the universe is in fact a dynamical quantity. The initial singularity problem then arose: if the universe is expanding it means that in the past it was much smaller than now. Letting the time run sufficiently backwards, it is possible to see all the galaxies collapsing to the same point. Although there is nothing in the past forcing the universe to expand as it does today, the most reliable cosmological models based on GR contained this controversial singularity, which will be named the Big Bang[3].

According to the Big Bang Theory (or Big Bang Model) the universe arises from a state of infinite matter/energy density. At that time, it was all concentrated in a region smaller than an atomic nucleus, but it started immediately to expand resulting, after billions of years of cosmological structure evolution, in the universe we see today. All the models including the Big-Bang singularity, differed only for the time behaviour of the expanding

[2]Except for the Steady State Cosmology which lasted until the discovery of the CMB.

[3]It is curious that the term Big Bang was first coined by Fred Hoyle in order to deride the theory (Hoyle supported the Steady State Cosmology).

radius, but they agreed on predicting an extremely isotropic background radiation created during the early hotter stages. This radiation, known as the Cosmic Microwave Background (CMB), was (accidentally) discovered back in 1967, providing the most important evidence of the initial Big Bang singularity.

After this milestone confirmation, cosmologists began to refine their theories on cosmic evolution. The result was the birth of a standard model of cosmology in which, after the Big Bang explosion, the universe underwent a radiation and then a matter dominated era. All the calculations needed for the universe to evolve from its initial stages to its final shape were performed within this framework. It was found that the genesis of all elements, galaxies, stars and all the other cosmological constituents, could fit with high precision in the predicted evolving background. Moreover, thanks to the newest technological progresses, physicists started to observe the very tiny (10^{-5}) CMB anisotropies, finding again strong evidences supporting the cosmological standard model.

It is then easy to realize how, except perhaps with some remaining issues regarding the very first instants (see chapter 4), towards the end of the twentieth century our understanding of the universe seemed to be close to complete.

1.2 The Present Cosmological Riddles

In the last twenty years cosmology has radically changed. It is by now well known that the cosmological standard model failed to explain some of the most recent astronomical observations. This has been the result of the rapid development of observational cosmology beginning in the 1990s. To date, it is established that the universe has undergone two unpredicted phases of cosmic acceleration and is pervaded by some yet undetected kind of matter which has been named *Dark Matter*. One of these two accelerated epochs is going on now, in the oldest age of our universe[4], and is due to an unknown form of energy, called *Dark Energy*. Conversely, the other phase took place during the very first moments after the Big Bang and is denoted with the name *Inflation*. In the following, we focus specifically on these top challenging issues of modern cosmology.

Inflation. We will analyse in more technical details this cosmological era in chapter 4, however some preliminary remarks can be anticipated here. The inflationary paradigm is perhaps the less dramatic among the modern cosmological problems. Inflation is postulated to have happened in a time

[4]This strange coincidence that the universe started to expand exactly during the human kind's living period, is sometimes regarded as an open problem of modern cosmology (Coincidence Problem).

when the energies involved were close to the Planck Scale (10^{19} GeV). Then it is not surprising that GR, together with the cosmological standard model, breaks down approaching the Big Bang. After all GR has been tested at relatively low energies and one can only assume it works also at higher energies, where, probably, some new (more fundamental) theory has to replace it. As a matter of fact, there are strong indications, especially from Black Hole physics where quantum and gravitational effects converge, that such a theory has to exist.

It follows from this that all the problems regarding this early accelerated epoch can be transferred from cosmology to fundamental physics. Our lack of predictivity during this cosmological phase can be attributed to the failure of GR in describing high energy physics. However a cosmological model can still be built in order to derive the right evolution and behaviour of the universe below these energies. This is what has been done during the last century in order to save the cosmological standard model.

In any case, a theory explaining how this primordial accelerated phase happened has been recently set up. It addresses the main inflationary paradigm problems, namely the Horizon and Flatness Problems (see chapter 4), and provides a mechanism to generate seeds for the late time cosmological structures. The model introduces a new scalar field, the *inflaton*, minimally coupled to Einstein's gravity. The inflaton not only forces the universe to expand, it also provides a way to create all matter fields of the particle physics Standard Model. This can be obtained exactly at the end of the accelerated phase, when the residual oscillations of the inflaton go on to excite the modes of other fields (Reheating). Unfortunately, the main reasons which lead us to consider this new scalar field are solely phenomenological. Despite its non-fundamental nature, this theoretical model of inflation, which we will denote "Standard Model of Inflation", succeeds in predicting all the data collected so far.

In conclusion the inflationary cosmological epoch is somehow produced by a high-energy modification of GR, whose effects can be modelled through the introduction of a scalar field. At the moment, this standard model of inflation is enough for deriving the right behaviour of the primordial expanding phase. As we will soon see, integrating this model into a framework able to solve the two other problems of modern cosmology, leads to a faithful representation of our universe.

Dark Energy. Dark Energy is certainly the most unexpected outcome of modern astronomical observations. Its discovery dates back to 1998, when a type Ia supernovae survey surprisingly found the universe accelerating, as later confirmed by CMB and gravitational lens experiments. Today the term Dark Energy denotes a strange form of energy filling the universe and driving its second accelerated phase. At first, physicists tried to identify Dark Energy with the quantum energy of vacuum, but this calculations predicts an energy density 10^{120} times different from the observed value. It was soon realized that the new expanding cosmic phase is driven by something completely different from usual physics.

The main feature of Dark Energy is negative pressure. The only way to make gravity repulsive is to consider a negative pressure fluid in the equations governing the cosmic evolution. This unnatural property cannot belong to any matter fields we discovered so far and the simplest way to obtain a late time cosmic acceleration[5], is to modify the gravitational field equations introducing the famous Einstein's cosmological constant Λ. Interestingly, the minimal modification introduced by setting a positive value for Λ into the cosmological equations, is enough to explain the contemporary universe acceleration.

The cosmological constant is not connected to any physical quantity and its introduction can only be justified by its accuracy to match astronomical data. In addition, we know that Dark Energy in general does not interact with other fields via any of the known forces of nature. As far as we have understood, Dark Energy requires a new kind of physics. This is probably what makes it so engaging in the eyes of the scientific community...

Dark Matter. Dark Energy is not the only surprise the universe has reserved for us. In the final quarter of the twentieth century[6], the increasing observations of galaxy dynamics suggested that most of the mass making up galaxies was missing. The anomaly was soon explained by postulating the existence of some undetected form of matter, namely Dark Matter. The particles composing Dark Matter do not interact electromagnetically, and the only way we can see their presence is through the gravitational effects they produce on the visible baryonic matter. First evidences for Dark Matter came from measuring the deviation it causes on galaxy rotations; later, CMB and gravitational lensing observations have confirmed these indications. By now these anomalies can be explained assuming that every galaxy is surrounded along the equatorial plane by a circular halo of Dark Matter.

Although we cannot detect it directly, we know that Dark Matter, unlike Dark Energy, shares some properties with ordinary matter. This leads to the idea that it is made of particles very similar to the baryonic ones. In fact, models where Dark Matter is composed by a yet undetected particle coming from some theoretical extension of the Standard Model of particle physics, are widely studied by physicists. Indeed, some of the best candidates for

[5]Theories which consider a scalar field to drive this second accelerated expansion, parallelling the role of inflaton in inflation, are also studied. They are collected under the name of *Quintessence*. However, invoking a scalar field does not seem to be so promising at late times as it is during inflation inasmuch as it implies other theoretical problems.

[6]Remarkably, in the 1930s the astrophysicist Fritz Zwicky already provided some evidences of missing galaxy mass, but it took forty years before someone else confirmed his predictions.

Dark Matter are supersymmetric partners of already discovered particles. In any case the Dark Matter is assumed to be a hypothetical particle which froze out from the cosmic plasma at some early times. It has then evolved, interacting with other matter gravitationally only, until it has reached the shape of a cosmic web, whose knots locate the large structure positions in the universe. As we can easily realise, Dark Matter played an important role during the cosmological period of structure formation. Hence, understanding its properties and dynamics will probably unveil some secrets our universe still hides.

1.3 The ΛCDM Model

Having pointed out the less penetrated cosmological issues, to understand what we know about the universe today, we just have to analyse the cosmic inventory, i.e. the abundance of different forms of energy in the universe. Our latest datasets coming from different sources, such as the CMB and supernovae surveys, seem to indicate that the energy budget of the universe is the following: 4% ordinary baryonic matter, 23% Dark Matter and 73% Dark Energy. This result indicates that what we have discovered so far is just a minimal fraction of the whole universe. There will be a huge amount of work for future cosmologists, astronomers and astrophysicists in order to analyse and understand this cosmic dark side. The experimental side, rather than the theoretical, seems perhaps to be more promising if we look at the immediate future. Hopefully, thanks to the incessant progress of technology, we will soon be able to build new devices to scan the sky. These improvements will permit us to observe the universe without "looking" at it. Instead of detecting the usual electromagnetic radiation, they will be able to measure elusive neutrinos or feeble gravitational waves, whose penetration will allow us to probe the universe up to the very first instants. These new observational windows will give us a large amount of data from which a renewed theoretical framework may emerge. In this scenario, the only thing on which we can bet, is that the universe has not run out of surprises.

Waiting for these forthcoming observations, we can only deal with theories facing the main problems of modern cosmology (see above). Putting together all the best models addressing those problems individually, we can obtain a general framework in which the behaviour of our universe is well described. This puzzle of different answers provides a faithful reproduction of the cosmic evolution and in its most reliable version takes the name of *ΛCDM Model* (Lambda-Cold Dark Matter). Its features are made explicit by its acronym: it assumes the existence of both a cosmological constant (Λ) and non-relativistic (cold) Dark Matter. Moreover, in order to reproduce the early time behaviour, the ΛCDM model is enlarged to include an inflationary primordial phase usually driven by a scalar field (inflaton).

This early accelerated stage is added, with its pros and cons, exactly in the form discussed previously (see chapter 4 for a deeper discussion). With this extension the ΛCDM model is the simplest to fit all the data collected so far.

We can briefly summarize the universe's history as given by the ΛCDM model as follows. Immediately after the Big Bang, which gave born to all the cosmic stuff (including spacetime itself), we can identify the so called Planck time (10^{-43} s after the BB). Before this moment we cannot make predictions with known physics. A unified comprehension of quantum and gravitational phenomena is indeed needed in order to push our knowledge closer to the Big Bang. In any case, from the Planck time on, we know all the main features of cosmic evolution. The first stage, beginning at Planck time and ending with Reheating (10^{-32} s), is the previously discussed inflationary phase. Baryogenesis, in which quarks were created (10^{-10} s), followed. Subsequently quarks started to aggregate together to form hadrons (10^{-4} s) which afterward formed the first light nuclei, i.e. H, D, T, He, ^{3}He and Li (100 s). Then we arrive at the matter-radiation equality (100,000 yrs) where the universe ceased to be radiation dominated and began to be matter dominated. After that an important event happened, namely Recombination (300,000 yrs), when the first atoms were formed and the CMB photons were released. This photon decoupling marks the time from which the universe ended to be opaque and started to be transparent to the electromagnetic radiation. The subsequent epoch, until the formation of galaxies, galaxy cluster, stars and all the other cosmological structures (few Gyrs), has been named the Cosmic Dark Age. The name is justified by the unusual lack of scientific observations regarding this era, though we think we understand how structures formed from the very homogeneous gas emerging from the Recombination, According to the ΛCDM model Dark Matter played a main role during the dark age. The real difference from all the other cosmological models lies exactly in these last cosmic phases.

In the ΛCDM model it is assumed that non-relativistic Dark Matter decoupled from all the other particles during some early stage. Thereafter its free (but for gravity) evolution determined the shape of the cosmological structures we see today. Remarkably, considering non-relativistic (cold) Dark Matter instead of relativistic (hot) or mixed Dark Matter matches the observations in a more consistent way. This, better than anything else, gives to the ΛCDM model the status of new standard cosmological theory.

Moreover, besides the issues regarding Dark Matter, central for the ΛCDM model is also the role of Dark Energy. The theory predicts indeed another transition from the matter dominated era to a Dark Energy dominated one ($\sim$5 Gyrs). In this new phase, in which we still live today (13.7 Gyrs), the universe expansion is accelerating again. In the ΛCDM model this is achieved by the progressive importance of the cosmological constant term in the gravitational field equations. Unfortunately, the model does not provide any information about the nature of this change and neither is able to explain why it happened. These very last features make the ΛCDM model more similar to a fitting of astronomical data than a self-consistent cosmological theory.

In conclusion, being a collection of individual solutions to different problems, the ΛCDM model lacks a solid theoretical foundation. Its strength relies mostly on its incredible accuracy of matching observational data. However, from a global point of view, this does not seem a satisfactory description of the universe we live in. Perhaps we missed somewhere some fundamental ingredient which could explain all the cosmological oddities. If such a sort of key exists, it is more probably hidden by our theoretical shortcoming. Perhaps even our theory of gravitation, despite its short-range achievements, fails to describe the universe at larger or higher-energy scales. Perhaps, at those scales, we need to modify GR.

1.4 Beyond General Relativity

During the first century of its life, numerous attempts, driven by different reasons, have been made to modify or replace GR. Even before the recent astronomical observations started to shake the minds of cosmologists, we register a large number of attempts, mainly led by theoretical arguments, to make some minimal or radical changes to the theory. Curiously enough, we can consider the famous introduction of the cosmological constant made by Einstein himself as the very first modification to GR. As it turned out, Einstein was wrong, showing that revising a successful and elegant theory as GR can be a hard game if played without any theoretical nor experimental justification[7].

Reasons for trying to alter GR however came out from the other great achievement of early twentieth century physics: Quantum Mechanics (QM). As soon as QM was set up in its final form during the 1920s, physicists began to formulate theories in which some or all of the new concepts coming from this new unusual theory were applied to GR. As a particularly successful example among these early works, we can pick out the so-called Einstein-Cartan(-Sciama-Kibble) theory, which tries to include the *spin*, a purely quantistic quantity, inside GR without spoiling general diffeomorphism invariance. Clearly, all these theories turned out to be either wrong, i.e. ruled out by experiments, or phenomenologically indistinguishable from GR itself, giving changes only at some untestable scales where GR cannot be expected to hold. This last kind of modification can however be helpful if we want

[7]Einstein's motivations for introducing the cosmological constant were more philosophical/theological than scientific. He was convinced that the universe had to be static and after the discovery of universe's expansion, he referred to this issue as "the biggest blunder of his life".

to unify GR with some other different theory. Leaving its well verified side as it is and changing its high energy limit has been considered as one of the most viable way to describe GR within a quantum field formalism.

Quantum Field Theory (QFT) is the natural evolution of QM. It is the most successful physical theory, experimentally speaking, which unifies, in a consistent theoretical way, QM and Special Relativity. QFT describes correctly the behaviour of all the non gravitational interactions (electromagnetic, weak and strong) and provides a unified framework where all of them are included in a single theoretical model, namely the Standard Model of particle physics, which has been tested with incredible accuracy[8]. It seems then natural that physicists started to wonder if also the excluded gravitational interaction could become part of this great unification. Consequently, the most invoked reason to modify GR has ever been this one: a consistent quantization of gravity, a theory called Quantum Gravity.

As was soon realized, the road to Quantum Gravity is a long, hard path and GR, as it is, cannot be easily quantized. Nowadays there is a large number of theories which have the aim to quantize gravity. This wide range spaces from Loop Quantum Gravity to String Theory, including Twistors, Causal Set, Causal Dynamical Triangulation and some others. None of them however, despite the huge amount of new concepts introduced, has yet succeeded in building a falsifiable quantum theory of gravity. This probably happens exactly because all the major modifications given by these theories arise in regions which experiments cannot probe. However, even though the search for Quantum Gravity has yet not given physical results of some relevance, it has stimulated the proliferation of alternative mathematical formulation of GR. Among these we can mention the tetrad (or vierbein) formalism and the Hamiltonian approach to GR (ADM formalism), which played an important role in the modern formulation of Quantum Gravity. The studies of all these new formalisms, besides the benefits given to theoretical physics, has provided also some hints in particular fields of mathematics such as mathematical physics and differential geometry. Moreover, having new ways of describing GR means new possibilities of modifying it. As a matter of fact, all the (quantum gravitational) theories cited above are constructed within some new formulation of gravity.

Although there are no possible phenomenological ways to verify them, there is still a promising framework where these theories can be theoretically tested: black hole physics. Near a black hole we can find together high spacetime curvature and short distances (which can be summarized as high energies) giving the perfect environment for gravitational quantum effects to emerge. In this scenario some small steps toward a unification between QFT and GR has already been made. Black hole thermodynamics is in

[8]A part from the yet undetected Higgs boson, which recently is the main target of LHC research.

fact a useful application of QFT on curved spacetime, which is nothing but the attempt of applying QFT on spaces curved by gravity. Remarkably, the Hawking's famous radiation, perhaps the most surprising result coming from black hole physics, has been first derived using this last approach. Now every theory of quantum gravity tries to explain somehow the black hole results, rather than furnish testable phenomenological observables. Indeed this is the best way we have to probe high energy modifications to gravity.

From all that we have said it emerges that theoretical arguments for modifying gravity abound and have been widely considered among physicists. Some kind of reconciliation between gravity and quantum physics seems to be needed in order to understand how the universe behaves in certain spacetime regions such as near the Big Bang or a black hole. The dream of finding this unified description of nature has been, and still is, the main boost in the research of gravity modifications.

Theoretical issues are not the only reasons for which physicists try to modify GR, even though they compose the dominant part. Philosophical and phenomenological motivations has been invoked as well. In the first of these two cases we find the already discussed Einstein's cosmological constant[9] and the well known Brans-Dicke Theory[10]. This last model represents an interesting attempt to build a gravitational theory compatible with Mach's principle. It substitutes the Newton constant with a dynamical scalar degree of freedom, or, to put it another way, it introduces a new scalar field non-minimally coupled to gravity. Brans-Dicke Theory has been considered for long time the most promising alternative to GR. Its generalized versions, the so called Scalar-Tensor Theories of gravitation (see section 3.1), are still today studied as possible models of inflation and Dark Energy.

The phenomenological side has instead been silent for long time. We must wait for the arrival of the recent cosmological problems to find some observations against GR predictions. However, once these observations were well consolidated, it has not taken long for physicists to start modifying gravity in order to account for experiments. As a matter of fact, the first modification due to phenomenological reasons was proposed immediately after the discovery of the galaxy rotation problem. This modification of gravity has not been made, as all later changes, on GR itself but, because of the nature of the problem to be solved, on Newtonian Gravity. The resulting theory, known as MOND (Modified Newtonian Dynamics), was created to modify the Newtonian potential in the slow acceleration regime. It succeeded in explaining the measured, and inconsistent with Newton theory, rotation rates of spiral galaxies, but it failed to account for the observations when applied to galaxy clusters. A relativistic version of MOND, known under the name of Tensor-Vector-Scalar Gravity (TeVeS), has been recently proposed

[9]See note at page 8.
[10]See page 27.

to explain also the post-Newtonian limit. Although this theory is considered a possible alternative to Dark Matter, it does not seem to be so promising as its competitor, which, as we have seen, has been included in the ΛCDM model.

More interesting attempts have come from the other two major issues of modern cosmology. As mentioned, the inflationary epoch can be attributed to some unknown behaviour of gravity at high energy scales. Accordingly, try to modify somehow GR in this limit to account for this needed primordial acceleration, might then be considered a natural way to pursue. This is exactly what physicists did, as soon as the inflationary problem arose, and what cosmologists still do today. The standard model of inflation is in fact nothing but a phenomenological modification to gravity which gives consistent deviations from GR only during the first instants of the universe. There is no need however to invoke a new scalar field. Modifications to gravitational interaction alone have been also found to be viable models for inflation. The success of the inflaton field model relies only on the accuracy it has to fit the astrophysical data collected so far. Nonetheless, other gravitational models could address specific questions in a more natural way or even work as a bridge between GR and high energy physics.

On the other side, the Dark Energy problem has been, in recent years, the most studied under the point of view of modifications of gravity. This has happened since the introduction of the cosmological constant, as it appears in the ΛCDM model and as pointed out previously, does not seem to have any theoretical background. An appropriate modification of the gravitational interaction can in this case lead to an accelerated late time expansion and simultaneously match the low energy limit of some high energy theory. However, every modified theory has to account for all the phenomenological tests GR brilliantly passed throughout its entire history. Since these experiments have been performed with very high precision and accuracy, all the considered modifications must be constrained to small deviations from GR itself, at least at those scales where GR has been well verified. This gives several difficulties to all modified theories inasmuch as they have to account for both the observed late time acceleration at large scales and the GR phenomenology at short scales. In any case these reasons have not stopped cosmologists from building models able to satisfy both behaviours. At the moment, these models, despite all the theoretical and phenomenological problems, are the best (and maybe the only) viable alternatives to the overused cosmological constant.

In conclusion, questioning over the possible modifications of gravity is not an useless exercise since, as it happened in the past, it could lead to a better understanding of the theories we trust and of the universe we live in. More than one hundred years ago scientists were trying to explain the small anomalies of Mercury's motion having no idea these could be described only by the great power of a theory such GR. Today we are trying to understand all the oddities we observe in the entire universe and no one knows what we are ignoring and going to discover. Will this (re)search lead to a new astonishing revolution in our vision of the universe? Time will tell...

Chapter 2

Lagrangian Formulation of General Relativity

In order to understand by which mechanisms it is possible to modify gravity, we first need to look at an alternative formulation of GR itself. Accordingly, this chapter is devoted to develope a lagrangian approach to the theory using both metric and Palatini principles. These formalisms, which will be introduced soon, will turn out useful later, when we will present how GR can be modified starting from chsnging its action.

The issues contained in the present chapter, conversely to the following ones, are exposed in a rather pedagogical way in order to clarify the techniques and the notation which will be massively utilized in what follows. No references are given through the text since the material presented can be found in some classical textbooks of GR (see for example [MTW73, Wal84]). For historical facts we refer to [FFR82] and the references therein.

2.1 Metric Variational Principle

We start presenting the most intuitive way to formulate GR from a variational principle, namely the metric formulation. Remarkable this method was one of the firsts which succesfully derived the correct gravitational field equations. Back in 1916, just five days before Einstein discovered exactly those equations, Hilbert succeeded in building a suitable action for gravity. Both the physicist and the mathematician found what we today call the Einstein field equations, eventually demonstraded to be the right equations describing gravity.

The method adopted by Hilbert is based on vary a suitable action for the gravitational field only with respect to the metric tensor $g_{\mu\nu}$. Such an action has to be invariant under general transformations of coordinates (diffeomorphisms) and has to depend in some way on the Riemann curvature tensor ${R_{\mu\nu\alpha}}^{\beta}$. This last tensor describes indeed the geometry of spacetime, which has to be connected with the distribution of mass by the field equations. It is then clear that any action constructed with the purpose of deriving these field equations, needs to treat somehow the Riemann tensor.

The unique scalar we can build from $R_{\mu\nu\alpha}{}^{\beta}$ with the only use of the metric tensor and which has only up to (linear) second-order derivatives in $g_{\mu\nu}$ (there are no invariants with only first derivatives), is the well-known curvature (Ricci) scalar R. As Hilbert intelligently understood, the first choice one can make for the gravitational action is then to consider simply R as the whole Lagrangians. We will see that this action alone suffices to derive the correctly Einstein field equations.

Consider then what we call the Einstein-Hilbert action

$$S_{EH} := \frac{1}{2}\int d^4x\sqrt{-g}R\,, \tag{2.1}$$

with $g := \det g_{\mu\nu}$. The integration is taken over the invariant measure of integration $d^4x\sqrt{-g}$ and the factor $1/2$ in front of the integral appears only to reproduce the correct Newtonian limit once derived the field equations. The curvature scalar R, Ricci tensor $R_{\mu\nu}$ and Riemann tensor $R_{\mu\nu\alpha}{}^{\beta}$ are defined by

$$R := g^{\mu\nu}R_{\mu\nu}\,; \qquad R_{\mu\nu} := R_{\mu\lambda\nu}{}^{\lambda}\,; \tag{2.2}$$

$$R_{\mu\nu\alpha}{}^{\beta} := \partial_\mu\Gamma^\beta_{\nu\alpha} - \partial_\nu\Gamma^\beta_{\mu\alpha} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - \Gamma^\beta_{\nu\lambda}\Gamma^\lambda_{\mu\alpha}\,; \tag{2.3}$$

where $\Gamma^\lambda_{\mu\nu}$ are the components of the usual Levi-Civita connection (Christoffel symbols), which are given in terms of the metric by

$$\Gamma^\lambda_{\mu\nu} := \frac{1}{2}g^{\lambda\sigma}\left(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right)\,. \tag{2.4}$$

We stress here that the Levi-Civita connection is the only connection compatible with the *metric condition*

$$\nabla_\lambda(g_{\mu\nu}) = 0\,, \tag{2.5}$$

with the *torsionless condition*

$$S_{\mu\nu}{}^{\lambda} = \Gamma^\lambda_{[\mu\nu]} = 0\,, \tag{2.6}$$

and it is uniquely defined once given a specific metric[1] $g_{\mu\nu}$. Here ∇ denotes the covariant derivative with respect to $\Gamma^\lambda_{\mu\nu}$, $S_{\mu\nu}{}^{\lambda}$ is called the torsion tensor[2] and $[\mu\nu]$ stands for antisimmetrization on the $\mu\nu$ indices. In particular, the torsionless condition (2.6) implies $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

---

[1]To find the unique expression of $\Gamma^\lambda_{\mu\nu}$, namely (2.4), from conditions (2.5), (2.6), we just have to subtract cyclic permutations of (2.5).

[2]It is a tensor being the difference of two connections.

We proceed considering a general variation of the action (2.1) with respect to the inverse metric tensor $g^{\mu\nu}$. We have[3]

$$\begin{aligned}\delta_g S_{EH} &= \frac{1}{2}\int d^4x\delta(\sqrt{-g}R) \\ &= \frac{1}{2}\int d^4x\left(R\delta\sqrt{-g}+\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}+\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}\right).\end{aligned} \tag{2.7}$$

In order to complete the variation we need to find $\delta\sqrt{-g}$ and $\delta R_{\mu\nu}$.

The first one depends on the variation of the metric determinant

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g\,. \tag{2.8}$$

To compute δg we need to recall from matrix theory the following expression holding for any matrix M and any differential operator δ

$$\delta(\ln\det M) = \frac{1}{\det M}\delta(\det M) = \mathrm{tr}(M^{-1}\delta M)\,, \tag{2.9}$$

where tr denotes the trace. Taking this equation with the metric matrix $g_{\mu\nu}$ gives

$$\delta g = g\,g^{\mu\nu}\delta g_{\mu\nu}\,. \tag{2.10}$$

Note that $\delta g^{\mu\nu}$ and $\delta g_{\mu\nu}$ are not connected by rising and lowering indices. They are two different tensors and not the covariant and controvariant part of the same tensor[4]. To find how they are connected we take the variation of $g^{\mu\nu}g_{\mu\nu}=4$, which gives

$$\delta g^{\mu\nu}g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} = 0\,. \tag{2.11}$$

Using this relation we then obtain the variation of the determinant of the metric

$$\delta g = -g\,g_{\mu\nu}\delta g^{\mu\nu}\,, \tag{2.12}$$

which inserted in eq. (2.8) gives

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{2.13}$$

We still need the variation of the Ricci tensor $R_{\mu\nu}$. This is explicitly given by

$$\begin{aligned}\delta R_{\mu\nu} &= \delta R_{\mu\lambda\nu}{}^{\lambda} \\ &= \partial_\nu\delta\Gamma^\lambda_{\lambda\mu} - \partial_\lambda\delta\Gamma^\lambda_{\mu\nu} + \delta\Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} \\ &\qquad +\Gamma^\alpha_{\mu\lambda}\delta\Gamma^\lambda_{\nu\alpha} - \delta\Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\lambda\alpha} - \Gamma^\alpha_{\mu\nu}\delta\Gamma^\lambda_{\lambda\alpha}\,,\end{aligned} \tag{2.14}$$

[3] We could equivalently vary with respect to $g_{\mu\nu}$, but the calculation would be more difficoult in that way.

[4] The misunderstanding raises from an abuse of notation. Denoting the two variations with $\delta g_{\mu\nu} = \delta[g_{\alpha\beta}]_{\mu\nu}$ and $\delta g^{\mu\nu} = \delta[g^{\alpha\beta}]^{\mu\nu}$, it is clear that $g^{\mu\lambda}g^{\nu\sigma}\delta[g_{\alpha\beta}]_{\mu\nu} = \delta[g_{\alpha\beta}]^{\mu\nu} \neq \delta[g^{\alpha\beta}]^{\mu\nu}$.

where the variation of Levi-Civita connection $\delta\Gamma^{\alpha}_{\mu\nu}$ has to be considered with respect to the metric $g_{\mu\nu}$. We can however avoid the calculation of $\delta\Gamma^{\alpha}_{\mu\nu}$ by noting that this quantity is indeed a tensor, despite the connection $\Gamma^{\alpha}_{\mu\nu}$ is not. This can be easily seen by the transformation law

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} \,. \tag{2.15}$$

Varying this equation with respect to $\Gamma^{\lambda}_{\mu\nu}$ (or equivalently with respect to $g^{\mu\nu}$) gives

$$\delta\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \delta\Gamma^{\alpha}_{\beta\gamma} \,, \tag{2.16}$$

since the inhomogeneous term vanishes in the variation. This tells us that $\delta\Gamma^{\lambda}_{\mu\nu}$ transforms exactly in a tensorial way, as was first realized by Palatini back in 1919[5]. Accordingly the covariant derivative of $\delta\Gamma^{\lambda}_{\mu\nu}$ is well defined and in particular we have

$$\nabla_{\nu}\delta\Gamma^{\lambda}_{\mu\lambda} = \partial_{\nu}\delta\Gamma^{\lambda}_{\mu\lambda} - \Gamma^{\alpha}_{\mu\nu}\delta\Gamma^{\lambda}_{\alpha\lambda} \tag{2.17}$$

$$\nabla_{\lambda}\delta\Gamma^{\lambda}_{\mu\nu} = \partial_{\lambda}\delta\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\lambda\alpha}\delta\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\lambda\mu}\delta\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\alpha}_{\lambda\nu}\delta\Gamma^{\lambda}_{\mu\alpha} \,. \tag{2.18}$$

Taking the difference of these two expressions and comparing it with eq. (2.14) we find[6]

$$\delta R_{\mu\nu} = \nabla_{\nu}\delta\Gamma^{\lambda}_{\mu\lambda} - \nabla_{\lambda}\delta\Gamma^{\lambda}_{\mu\nu} \,. \tag{2.19}$$

This is called *Palatini identity*.

We can now finish to derive the equation of motion for the Einstein-Hilbet action. Substituting eqs. (2.13) and (2.19) back in eq. (2.7) we get

$$\begin{aligned} \delta_g S_{EH} &= \frac{1}{2}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} \\ &\quad + \frac{1}{2}\int d^4x\sqrt{-g}\left[\nabla_{\nu}\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right) - \nabla_{\lambda}\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right)\right] \,, \end{aligned} \tag{2.20}$$

where we have used the metric condition (2.5) to shift $g^{\mu\nu}$ inside the covariant derivatives. The last two terms are just total derivatives. Considering vanishing boundary conditions (all the physical fields and their derivatives are zero on the boundaries of the integration region) these can be integrated away using Stokes theorem[7]. The variational principle $\delta_g S_{EH} = 0$ gives thus

[5]The original article (in italian) is [Pal19].

[6]A faster, but not covariant, way to perform this calculation is to evaluate $\delta R_{\mu\nu}$ in the normal frame, where all the Γ's vanish, and then apply the minimal coupling principle, namely replace all the normal derivatives with the covariant ones. Using this procedure eq. (2.19) follows immediately from eq. (2.14).

[7]Actually this is not true for gravity where we cannot impose vanishing boundary conditions. However, adding a total divergence to the action, namely the Gibbons-Hawking-York boundary term [GH77, Yor72], is still possible to have a well-define variational procedure.

the following equations of motion

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0\,, \tag{2.21}$$

which are nothing but the Einstein field equations in vacuum. $G_{\mu\nu}$ is called the Einstein tensor and is used to denote the left hand side of the gravitational field equation.

Finally, we note that if we want to include a cosmological constant Λ in the theory, we need just to add to the Einstein-Hilbert action the following constant term

$$S_\Lambda := -\int d^4x \sqrt{-g}\, 2\Lambda\,. \tag{2.22}$$

The variation of the whole action will give the correct modified Einstein tensor with a cosmological constant

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\,. \tag{2.23}$$

2.2 The Matter Action

In order to obtain the general Einstein equations from a variational principle, we need to extend the Einstein-Hilbert action (2.1) with a new term giving the right hand side of the fields equation. This is nothing but the action of all the physical fields but the gravitational one. It is called the matter action S_M and it depends, besides all the other fields collectively denoted by Ψ, on the gravitaional field $g_{\mu\nu}$, in order to assure the invariance under general transformations of coordinates[8].

The whole action we must consider is then

$$\begin{aligned} S &= S_{EH}(g_{\mu\nu}) + S_M(g_{\mu\nu}, \Psi) \\ &= \int d^4x \sqrt{-g} \left[\frac{1}{2} R + \mathcal{L}_M(g_{\mu\nu}, \Psi) \right]\,, \end{aligned} \tag{2.24}$$

where $\mathcal{L}_M$ is the matter Lagrangian. The field equations are derived varying with respect to $g_{\mu\nu}$ and Ψ. Since S_{EH} does not depend on any other field besides $g_{\mu\nu}$, the variation with respect to Ψ will give the usual equations of motion for all the other physical fields,

$$\frac{\delta S_M}{\delta \Psi} = 0\,. \tag{2.25}$$

[8]In general the matter action can depend also on the connection $\Gamma^\lambda_{\mu\nu}$, which is however, at least in the metric formulation, a function of the metric $g_{\mu\nu}$. In the next section, when the connection will be considered independent from the metric, this issue will be briefly discussed, as it can imply deviations from canonical GR.

Varying with respect to $g_{\mu\nu}$ gives the same result we found in the previous section for the Einstein-Hilbert action

$$\delta_g S_{EH} = \frac{1}{2}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}\,, \tag{2.26}$$

plus the variation of the matter action

$$\begin{aligned} \delta_g S_M &= \int d^4x\,(\delta\sqrt{-g}\mathcal{L}_M + \sqrt{-g}\delta\mathcal{L}_M) \\ &= \int d^4x\sqrt{-g}\left(\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_M\right)\delta g^{\mu\nu}\,, \end{aligned} \tag{2.27}$$

where we have used eq. (2.13) to compute $\delta\sqrt{-g}$. The variational principle applied on the whole S gives then

$$\delta S = \int d^4x\sqrt{-g}\left(\frac{1}{2}G_{\mu\nu} + \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_M\right)\delta g^{\mu\nu} = 0\,. \tag{2.28}$$

We now give the general definition of the energy-momentum tensor as

$$T_{\mu\nu} := g_{\mu\nu}\mathcal{L}_M - 2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}\,. \tag{2.29}$$

This energy-momentum tensor is exactly the same of the corresponding one we can derive using Noether theorem with diffeomorphism invariance of the action. In other words, it has the same meaning of the usual $T_{\mu\nu}$ we can find in all the other formulation of GR. The equations we obtain from the variation (2.28) are then the standard Einstein field equations, which read

$$G_{\mu\nu} = T_{\mu\nu}\,. \tag{2.30}$$

2.3 Palatini Variational Principle

We present now another way of deriving the gravitational field equations, namely the Palatini method of variation. This formulation has the purpose of relaxing some of the assumptions the metric approach relies on. It consists in considering the connection entering the Einstein-Hilbert action as completely independent from $g_{\mu\nu}$, though it is still taken to be torsion free[9].

A variation with respect to the new degrees of freedom of the independent connection will produce new equations of motion, which will be used to determine the relation between the Levi-Civita connection and the introduced independent connection. The Einstein field equations are in fact solely defined by the use of the Levi-Civita connection and cannot be considered as the physical equations of the gravitational field if constructed with the use of a different connection. The relation between the two connections

[9] A connection is said to be torsion free if it satisfies the torsionless condition (2.6).

is needed for replacing the unphysical connection in favor of the physical Levi-Civita connection. As we will see, the Einstein-Hilbert action alone will imply that the two connections are indeed the same, leading, again, to the yet derived Einstein field equation.

Consider the Einstein-Hilbert action plus the matter action. We will denote with $\hat{\Gamma}^{\lambda}_{\mu\nu}$ the new independent torsionless connection to dinstinguish it from the usual Levi-Civita connection $\Gamma^{\lambda}_{\mu\nu}$. Moreover all the quantities now formed with $\hat{\Gamma}^{\lambda}_{\mu\nu}$ will be in general marked with an overhat. For example, the curvature scalar and Ricci tensor constructed with $\hat{\Gamma}^{\lambda}_{\mu\nu}$ will be denoted by $\hat{R}$ and $\hat{R}_{\mu\nu}$, respectively; while the same tensors will be as usual denoted by R and $R_{\mu\nu}$ if composed by the Levi-Civita connection. The action reads then

$$\begin{aligned} S &= S_{EH}(g_{\mu\nu}, \hat{\Gamma}^{\lambda}_{\mu\nu}) + S_M(g_{\mu\nu}, \Psi) \\ &= \int d^4x\sqrt{-g}\left[\frac{1}{2}\hat{R} + \mathcal{L}_M(g_{\mu\nu}, \Psi)\right] . \end{aligned} \tag{2.31}$$

The matter action does not depend on the independent connection $\hat{\Gamma}^{\lambda}_{\mu\nu}$. This is a specific postulate of Palatini variational method. Allowing the action for this dependence will lead to yet another formulation, namely the metric-affine variation, which in general implies some deviation from GR itself. We will not discuss the metric-affine variational principle since it would lead us toward arguments completely dissociated from the main scopes of this work[10]. We remain inside the framework of Palatini variation, considering the matter action indipendent from $\hat{\Gamma}^{\lambda}_{\mu\nu}$.

Varying the action (2.31) with respect to $g^{\mu\nu}$ gives

$$\delta_g S = \frac{1}{2}\int d^4x\sqrt{-g}\left(\hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} - T_{\mu\nu}\right)\delta g^{\mu\nu} , \tag{2.32}$$

from which we read off the following equations of motion[11]

$$\hat{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\hat{R} = T_{\mu\nu} . \tag{2.33}$$

Note that in the Palatini approach the Ricci tensor is a function of $\hat{\Gamma}^{\lambda}_{\mu\nu}$ alone. This means that the boundary terms coming from the variation of $R_{\mu\nu}$ with respect to $g_{\mu\nu}$ we had in the metric formulation, no longer appears here. They belong to the variation with respect to $\hat{\Gamma}^{\lambda}_{\mu\nu}$ now. We also stress that these are not the Einstein field equation (2.30). Although the form of the equations is the same, the connection enter in eq. (2.33) is not the Levi-Civita connection.

[10] For the reader interested in this subject we cite [HK78, HLS81].

[11] Note that $\hat{R}_{\mu\nu}$ has not to be symmetric as $R_{\mu\nu}$. This happens because the Riemann tensor has less symmetries if the connection is not metric.

The variation with respect to the independent connection $\hat{\Gamma}^{\lambda}_{\mu\nu}$ gives

$$\begin{aligned}\delta_\Gamma S &= \int d^4x\sqrt{-g}\,g^{\mu\nu}\delta\hat{R}_{\mu\nu} \\ &= \int d^4x\sqrt{-g}\,g^{\alpha\beta}\left(\hat{\nabla}_\lambda\delta\hat{\Gamma}^{\lambda}_{\alpha\beta}-\hat{\nabla}_\beta\delta\hat{\Gamma}^{\lambda}_{\alpha\lambda}\right), \qquad (2.34)\end{aligned}$$

where we used the Palatini identity (2.19) to compute $\delta\hat{R}_{\mu\nu}$ and denoted by $\hat{\nabla}$ the covariant derivative of $\hat{\Gamma}^{\lambda}_{\mu\nu}$. Integrating by parts we get

$$\begin{aligned}\delta_\Gamma S &= \int d^4x\hat{\nabla}_\lambda\left(\sqrt{-g}\,g^{\alpha\beta}\delta\hat{\Gamma}^{\lambda}_{\alpha\beta}\right)-\int d^4x\hat{\nabla}_\beta\left(\sqrt{-g}\,g^{\alpha\beta}\delta\hat{\Gamma}^{\lambda}_{\alpha\lambda}\right) \\ &\quad -\int d^4x\left[\hat{\nabla}_\lambda\left(\sqrt{-g}\,g^{\alpha\beta}\right)\delta\hat{\Gamma}^{\lambda}_{\alpha\beta}-\hat{\nabla}_\beta\left(\sqrt{-g}\,g^{\alpha\beta}\right)\delta\hat{\Gamma}^{\lambda}_{\alpha\lambda}\right]. \qquad (2.35)\end{aligned}$$

In order to proceed the calculation we must recall the definition of *tensor density* and how covariant derivatives act on it. A tensor density is a quantity which transforms as a tensor except for extra multiplicative factors of the Jacobian. The number of times the Jacobian is multiplied is called the weight of the tensor density. For example, a (second-rank) tensor density $\mathcal{F}^{\mu}{}_{\nu}$ of weight W transforms as

$$\mathcal{F}'^{\mu}{}_{\nu}=J^W\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\mathcal{F}^{\alpha}{}_{\beta}\,, \qquad (2.36)$$

where J is the Jacobian. An important property is that if $\mathcal{F}^{\mu\nu\ldots}_{\alpha\beta\ldots}$ is a tensor density of weight W, then $(-g)^{W/2}\mathcal{F}^{\mu\nu\ldots}_{\alpha\beta\ldots}$ is an ordinary tensor[12]. We can thus see that all the terms covariantly derived in eq. (2.35) are indeed tensor densities of weight -1, since multiplication by $(-g)^{-1/2}$ gives back ordinary tensors. It still remains to know how covariant derivatives act on tensor densities. Requiring that the usual properties of differential operators hold and that the result is still a tensor density of the same weight of the starting one, the covariant derivatives have to act on tensor densities as

$$\nabla_\rho\mathcal{F}^{\alpha\beta\ldots}_{\mu\nu\ldots}:=(-g)^{-\frac{W}{2}}\nabla_\rho\left[(-g)^{\frac{W}{2}}\mathcal{F}^{\alpha\beta\ldots}_{\mu\nu\ldots}\right], \qquad (2.37)$$

where in the right hand side the covariant derivative act as usual on ordinary tensor. Using this definitions back in eq. (2.35) we get

$$\begin{aligned}\delta_\Gamma S &= \int d^4x\sqrt{-g}\,\hat{\nabla}_\lambda\left(g^{\alpha\beta}\delta\hat{\Gamma}^{\lambda}_{\alpha\beta}\right)-\int d^4x\sqrt{-g}\,\hat{\nabla}_\beta\left(g^{\alpha\beta}\delta\hat{\Gamma}^{\lambda}_{\alpha\lambda}\right) \\ &\quad -\int d^4x\sqrt{-g}\left[\left(\hat{\nabla}_\lambda g^{\alpha\beta}\right)\delta\hat{\Gamma}^{\lambda}_{\alpha\beta}-\left(\hat{\nabla}_\beta g^{\alpha\beta}\right)\delta\hat{\Gamma}^{\lambda}_{\alpha\lambda}\right]. \qquad (2.38)\end{aligned}$$

[12]This follows from the summing properties of tensor densities (weights add up) and the fact that $\sqrt{-g}$ is a tensor density of weight -1.

Now the first two terms are nothing but total derivatives (with respect to the invariant measure of integration $d^4x\sqrt{-g}$) which vanish thanks to the Stokes theorem[13]. Hence the variation with respect to $\hat{\Gamma}^{\lambda}_{\mu\nu}$ gives in the end

$$\delta_{\Gamma} S = \int d^4x\sqrt{-g}\left(\hat{\nabla}_{\lambda} g^{\alpha\beta} - \delta^{\beta}_{\lambda}\hat{\nabla}_{\sigma} g^{\alpha\sigma}\right)\delta\hat{\Gamma}^{\lambda}_{\alpha\beta} = 0\,, \tag{2.39}$$

which implies

$$\hat{\nabla}_{\lambda} g^{\alpha\beta} - \delta^{(\beta}_{\lambda}\hat{\nabla}_{\sigma} g^{\alpha)\sigma} = 0\,, \tag{2.40}$$

where $(\alpha\beta)$ denotes symmetrization with respect to the indeces $\alpha\beta$. We have to take only the symmetric part since $\hat{\Gamma}^{\lambda}_{\mu\nu}$ is symmetric in $\mu\nu$ (torsionless). Eq. (2.40) is a linear system of 40 equations in the 40 variables $\hat{\nabla}_{\lambda} g^{\mu\nu}$. It admits the only solution[14]

$$\hat{\nabla}_{\lambda} g^{\mu\nu} = 0\,, \tag{2.41}$$

which implies

$$\hat{\nabla}_{\lambda} g_{\mu\nu} = 0\,. \tag{2.42}$$

This is nothing but the metric condition (2.5) for $\hat{\Gamma}^{\lambda}_{\mu\nu}$. Since $\hat{\Gamma}^{\lambda}_{\mu\nu}$ satisfies both the torsionless and metric conditions, it has to coincides with the Levi-Civita connection. In brief, this means

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}\,. \tag{2.43}$$

The gravitational field equations we have found in the Palatini variation (eq. (2.33)) becomes then the canonical Einstein field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}\,. \tag{2.44}$$

The theory is then physically the same using either metric or Palatini variations. The gravitational field equations are indeed the Einstein field equations in both theories. The major difference is that in the Palatini formulation the metric condition is derived from a variational principle rather than be imposed a priori as in the metric approach. This means that we have a postulate less to impose when we decide to take the Palatini variation.

Moreover the Palatini formulation is easier to compare with the well studied theories of particle physics, where a gauge field (the connection) is needed to describe interacting fields. Describing gravity as a gauge theory will probably leads towards a canonical quantization. However, all the details of this relation are far to be understood and still today the research of the so called quantum gravity can be regarded as a work in progress.

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[13] See note at page 16.

[14] A quick way to find the solution is to take the contraction between λ and β (or λ and α) which gives $\hat{\nabla}_{\lambda} g^{\alpha\lambda} = 0$. Substituting this back in eq. (2.40) we obtain immediately condition (2.41).

Chapter 3

Modified Actions for Gravity

The present chapter is devoted to introduce some of the most popular theories with the aim of modifying GR: Scalar-Tensor Theories and $f(R)$ Theories. First we will treat Scalar-Tensor theories in both metric and Palatini formulation and we will mention a special case: Brans-Dicke theories. Then we will analyze $f(R)$ gravity showing how, in both metric and Palatini approaches, it can be recasted into the form of a particular Brans-Dicke theory. Finally we will present an unified way to formulate these theories, which will be named Generalized (Modified) Gravity. This class of theories will contain as subclasses both Scalar-Tensor and $f(R)$ gravity. Thus the result we will obtain within generalized gravity will hold automatically for these other theories.

In general for each theory we present the action and derive the equations of motion. Since doing this every time could become a repetitive work, we will discuss several details and remarks only in the final section, where we will treat the generalized approach to modified gravity. In any case, it should be stressed that all the remarks pointed out there hold also for the theories discussed previously, inasmuch as they can be seen as particular realizations of generalized gravity. References to the presented material will be given troughout the text.

3.1 Scalar-Tensor Theories

In this section we encounter our first modification of the Einstein-Hilbert action (2.1), which consists in introducing a new scalar field ϕ generally coupled to gravity. The coupling is obtained by an arbitrary function of ϕ which interacts linearly with the curvature scalar R, whilst its (generalised) kinetic and potential terms are added as usual. This class of theory goes under the name of *Scalar-Tensor Theories of gravity.* Motivations for introducing such scalar field can be easily found in the literature. As a matter of fact, the well-known Brans-Dicke Theory [BD61] and the present standard model of inflation (see chapter 4) are specific examples within this class of theories. In the following we derive field equations for the general Scalar-Tensor theory in both metric and Palatini approaches and briefly discuss the Brans-Dicke case. For further details and applications we refer to [FM03, Far04].

Metric Formulation

The general action for Scalar-Tensor Gravity reads

$$S_{ST} := \int d^4x\sqrt{-g}\left[\frac{1}{2}F(\phi)R - \frac{\omega(\phi)}{2}(\partial\phi)^2 - V(\phi) + \mathcal{L}_M\right] , \qquad (3.1)$$

where $(\partial\phi)^2 := g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. R and $\mathcal{L}_M$ are the usual curvature scalar and matter Lagrangian, respectively. $F(\phi)$, $\omega(\phi)$ and $V(\phi)$ are taken to be arbitrary general functions of the scalar field ϕ alone. In the metric approach the field equations are obtained varying action (3.1) independently with respect to the (inverse) metric tensor $g^{\mu\nu}$ and the scalar field ϕ.

The metric variation gives the following gravitational field equations

$$F\,G_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = T^{(\phi)}_{\mu\nu} + T_{\mu\nu}\,, \qquad (3.2)$$

where $G_{\mu\nu}$ is the common Einstein tensor and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. $T_{\mu\nu}$ is the energy-momentum tensor derived from the matter Lagrangian $\mathcal{L}_M$ as given by (2.29) and $T^{(\phi)}_{\mu\nu}$ is the scalar field energy-momentum tensor defined by

$$T^{(\phi)}_{\mu\nu} := \omega\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}[\frac{1}{2}\omega(\partial\phi)^2 + V(\phi)]\,. \qquad (3.3)$$

Eq. (3.2) is clearly different from the canonical Einstein field equations (2.30). The coupling function $F(\phi)$ and its (covariant) derivatives appear explicitly in the right hand side, while the scalar field energy-momentum tensor is added to the usual $T_{\mu\nu}$. We note that the new gravitational coupling, rather than being constant as in canonical GR (the Newton constant), is now parametrized by ϕ through the function F. This means that the gravitational "constant" depends now, through the scalar field ϕ, by the spacetime event considered. This is more explicit if we rewrite eq. (3.2) as

$$G_{\mu\nu} = \frac{1}{F}\left(T^{(\phi)}_{\mu\nu} + T_{\mu\nu} + T^{(F)}_{\mu\nu}\right)\,, \qquad (3.4)$$

with

$$T^{(F)}_{\mu\nu} := \nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F\,. \qquad (3.5)$$

Variation of action (3.1) with respect to the scalar field ϕ gives the following modified Klein-Gordon (KG) equation

$$\Box\phi + \frac{1}{2\omega}\left(\omega_{,\phi}\nabla^\mu\phi\nabla_\mu\phi - 2V_{,\phi} + F_{,\phi}R\right) = 0\,, \qquad (3.6)$$

where $_{,\phi}$ means differentiation with respect to ϕ. We note that choosing $F = \omega = 1$, in which case action (3.1) becomes the Einstein-Hilbert action minimally coupled with the canonical scalar field ϕ, eq. (3.2) and eq. (3.6) turn out to be the GR Einstein field equations (with a scalar matter field energy-momentum tensor) and the usual KG equation

$$\Box\phi - V_{,\phi} = 0\,. \tag{3.7}$$

The equation of motion for the scalar field (3.6) turns out to be coupled with the gravitational field equations (3.2) also through R, instead of being coupled only via $g_{\mu\nu}$. However we can easily eliminate R in eq. (3.6) considering the trace of eq. (3.2) which gives

$$R = \frac{1}{F}\left(3\Box F - T^{(\phi)} - T\right)\,, \tag{3.8}$$

where $T^{(\phi)}$ and T are the traces of $T^{(\phi)}_{\mu\nu}$ and $T_{\mu\nu}$, respectively. In this manner, once given the metric tensor, the (modified) KG are decoupled from the modified Einstein equations.

Palatini Formulation

It is also possible to treat Scalar-Tensor gravity within the Palatini approach. Finding equations of motion is more complicated in the Palatini case than in the metric case. This is due to the simultaneous variation with respect to the metric $g^{\mu\nu}$ and the independent (torsionless) connection $\hat{\Gamma}^{\lambda}_{\mu\nu}$. First of all we must rewrite action (3.1) underlining the explicit dependences by $g_{\mu\nu}$ or $\hat{\Gamma}^{\lambda}_{\mu\nu}$. We have

$$S_{ST} := \int d^4x\sqrt{-g}\left[\frac{1}{2}F(\phi)\hat{R} - \frac{\omega(\phi)}{2}(\partial\phi)^2 - V(\phi) + \mathcal{L}_M(g_{\mu\nu},\Psi)\right]\,, \tag{3.9}$$

where now the curvature scalar is composed solely by the independent connection $\hat{\Gamma}^{\lambda}_{\mu\nu}$ and marked with an overhat[1]: $\hat{R} := g^{\mu\nu}\hat{R}_{\mu\nu}$. The matter Lagrangian does not depend on $\hat{\Gamma}^{\lambda}_{\mu\nu}$ according to the Palatini prescription.

An independent variation with respect to the metric $g^{\mu\nu}$, the connection $\hat{\Gamma}^{\lambda}_{\mu\nu}$ and the scalar field ϕ gives, respectively,

$$F\left(\hat{R}_{(\mu\nu)} - \frac{1}{2}\hat{R}\,g_{\mu\nu}\right) = T^{(\phi)}_{\mu\nu} + T_{\mu\nu}\,; \tag{3.10}$$

$$\hat{\nabla}_{\mu}\left(\sqrt{-g}g^{\alpha\beta}F\right) - \delta^{(\beta}_{\mu}\hat{\nabla}_{\sigma}\left(\sqrt{-g}g^{\alpha)\sigma}F\right) = 0\,; \tag{3.11}$$

$$\hat{\nabla}^{\mu}\hat{\nabla}_{\mu}\phi + \frac{1}{2\omega}\left(\omega_{,\phi}\partial^{\mu}\phi\partial_{\mu}\phi - 2V_{,\phi} + F_{,\phi}\hat{R}\right) = 0\,. \tag{3.12}$$

[1]See section 2.3.

As we did in the metric case we can substitute $\hat{R}$ in eq. (3.12) as given from the trace of eq. (3.10)

$$\hat{R} = -\frac{T^{(\phi)} + T}{F}\,, \tag{3.13}$$

in order to decouple the KG equation. Moreover, in analogy to eq. (2.40), eq. (3.11) is a system of 40 equations in the 40 variables $\hat{\nabla}_\mu(\sqrt{-g}g^{\alpha\beta}F)$ which admits the unique solution

$$\hat{\nabla}_\mu\left(\sqrt{-g}g^{\alpha\beta}F\right) = 0\,. \tag{3.14}$$

At this point we define a new metric $h_{\mu\nu}$ conformally connected to $g_{\mu\nu}$ with conformal factor F

$$h_{\mu\nu} := F\, g_{\mu\nu}\,. \tag{3.15}$$

In terms of this new metric condition (3.14) reads

$$\hat{\nabla}_\mu\left(\sqrt{-h}\, h^{\alpha\beta}\right) = 0\,, \tag{3.16}$$

which implies

$$\hat{\nabla}_\mu h_{\alpha\beta} = 0\,. \tag{3.17}$$

This is nothing but the metric condition (2.5) for the connection $\hat{\Gamma}^\lambda_{\mu\nu}$ with respect to the metric $h_{\mu\nu}$. Since $\hat{\Gamma}^\lambda_{\mu\nu}$ was assumed to be torsionfree from the beginning, it turns out to be the Levi-Civita derivative with respect to $h_{\mu\nu}$

$$\hat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}h^{\lambda\sigma}(h_{\nu\sigma,\mu} + h_{\mu\sigma,\nu} + h_{\mu\nu,\sigma})\,, \tag{3.18}$$

which we can also rewrite in terms of F and $g_{\mu\nu}$ as

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2F}\left[2\delta^\lambda_{(\mu}\partial_{\nu)}F + g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma F\right]\,, \tag{3.19}$$

with $\Gamma^\lambda_{\mu\nu}$ denoting the usual Levi-Civita connection formed with $g_{\mu\nu}$. This result tells us that the independent connection $\hat{\Gamma}^\lambda_{\mu\nu}$ can be substituted whenever it appears in our equations in favour of $\Gamma^\lambda_{\mu\nu}$. In particular eqs. (3.10) and (3.12) become

$$\begin{aligned}&F\,G_{\mu\nu} + \frac{3}{2F}(\nabla_\mu F)(\nabla_\nu F) - \nabla_\mu\nabla_\nu F\\ &-\frac{1}{2}g_{\mu\nu}\left[(1-\frac{3}{F})\nabla_\sigma\nabla^\sigma F + \frac{3}{2F}(\nabla_\sigma F)(\nabla^\sigma F)\right] = T^{(\phi)}_{\mu\nu} + T_{\mu\nu}\,;\end{aligned} \tag{3.20}$$

$$\Box\phi - \frac{1}{3F}\partial^\mu\phi\partial_\mu F + \frac{1}{2\omega}\left[\omega_{,\phi}\partial^\mu\phi\partial_\mu\phi - 2V_{,\phi} - \frac{F_{,\phi}}{F}(T^{(\phi)} + T)\right] = 0\,. \tag{3.21}$$

The modified gravitational field equations (3.20) can be written, as we did in the metric case, in the (canonical) form of eq. (3.4) defining a new energy-momentum tensor as

$$\begin{aligned} T^{(F)}_{\mu\nu} \;:=\; & -\frac{3}{2F}(\nabla_\mu F)(\nabla_\nu F) + \nabla_\mu \nabla_\nu F \\ & +\frac{1}{2} g_{\mu\nu}\left[(1-\frac{3}{F})\nabla_\sigma \nabla^\sigma F + \frac{3}{2F}(\nabla_\sigma F)(\nabla^\sigma F)\right] . \end{aligned} \tag{3.22}$$

which depends by ϕ through the function F.

If we compare the Palatini equations of motion (3.20) and (3.21) with the corresponding metric equations (3.2) and (3.6), or better the two energy-momentum tensors $T^{(F)}_{\mu\nu}$ given by eqs. (3.5) and (3.22), we notice that in the Palatini formulation of the theory more terms, depending on the modifying function F, appear. This means that, unlike in GR, the two formalisms do not lead to the same theory. They imply different equations of motion and consequently different physical results. This is an important point meaning that the two approaches are not just two distinguished mathematical formulations of the same theory, but correspond indeed to two different physical theories.

Brans-Dicke Theory

The most famous example of Scalar-Tensor Theory is Brans-Dicke Theory. We briefly discuss here some aspects of this theory since it will be of interest later on. The following analysis is performed uniquely in the metric approach since this is the natural and more studied formulation of the theory, and, consequently, the results we need have been obtained whitin this framework.

In Brans-Dicke Theory we set[2]

$$F(\phi) = \phi \qquad \text{and} \qquad \omega(\phi) = \frac{\omega_0}{\phi}\,, \tag{3.23}$$

with the constant ω_0 called the BD parameter. The Brans-Dicke action reads then

$$S_{BD} := \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi R - \frac{\omega_0}{2\phi}(\partial\phi)^2 - V(\phi) + \mathcal{L}_M(g_{\mu\nu},\Psi)\right] . \tag{3.24}$$

Since the BD parameter multiply the kinetic term of the scalar field, it plays an important role in the theory and many possible values of it has been considered in the past. As we will see later, it will be of crucial importance in identify different modified gravity theories.

[2]The first model proposed in [BD61] considered also $V(\phi) = 0$. However, for our purposes, the potential is taken arbitrary.

Independent variation with respect to $g^{\mu\nu}$ and ϕ gives the following equations of motion

$$G_{\mu\nu} = \frac{1}{\phi} T^{(tot)}_{\mu\nu} \,; \tag{3.25}$$

$$\frac{2\omega_0}{\phi}\Box\phi - \frac{\omega_0}{\phi^2}(\partial\phi)^2 - 2V_{,\phi} + R = 0 \,; \tag{3.26}$$

with $T^{(tot)}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T_{\mu\nu} + T^{(F)}_{\mu\nu}$, and where in this case

$$T^{(F)}_{\mu\nu} = \nabla_\mu \nabla_\nu \phi - g_{\mu\nu}\Box\phi \,. \tag{3.27}$$

From eq. (3.25) is evident the purpose of the original Brans-Dicke model [BD61]. The gravitational coupling depends now on the scalar field ϕ. This is exactly the mechanism they found for building a gravitational theory respecting the Mach's principle.

Brans-Dicke theory has been considered the most promising alternative to GR for long time. This means that we can find a huge amount of works in the literature which treated the theory under several point of views. In particular for what concerns our aims, some of the theoretical results already derived for this theory can be also applied to other theories which can somehow be regarded as equivalents to Brans-Dicke Theory.

3.2 Higher-order Gravity

Introducing a scalar field in the action is not the only way to modify gravity. However, considering other kinds of field, such as spinor or vector fields, can be regarded as a generalization of the scalar field idea with the only intention to complicate things. Models where new vectors and tensors couple to gravity has been well studied (TeVeS for example), but the main cosmological observations can usually be better explained within the simpler scalar field modification or within the ΛCDM model frameworks. For this reason we will not discuss introductions of other kinds of field in the remaining part of this work.

A possible and straightforward modification of GR, without resort to other fields, consists to include in the Einstein-Hilbert action more curvature invariants in addition to R. These invariants are solely constructed with $g_{\mu\nu}$ and its derivatives and do not rely on any other field. They can be derived contracting the Riemann tensor in every possible way. For instance we can form the invariants $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, or even consider strange contractions such as[3] $R^{\mu\nu\alpha\beta}R_{\mu\alpha}R_{\nu\beta}$. Invariants constructed with the Weyl tensor $C_{\mu\nu\alpha\beta}$ can also be considered, such as $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$. In general all

[3] The combination $R^{\mu\nu\alpha\beta}R_{\mu\nu}R_{\alpha\beta}$ vanishes because of the symmetry properties of the Riemann tensor.

the possible contractions of the Riemann tensor with itself and the metric $g_{\mu\nu}$ can be considered in building such invariants. These invariants however introduce in the action more derivatives with respect to the metric tensor and for this reason the resulting theories are called higher-order theories of gravity.

A specific combination one can makes is the so called Gauss-Bonnet invariants

$$\mathcal{G} := R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\,. \tag{3.28}$$

Models considering this invariant in order to create some deviations from GR are collected under the name of Gauss-Bonnet Gravity. This class of theories is expecially studied as a possible low energy limit of String Theory (see for example [Tse96]). The most important feature of the scalar $\mathcal{G}$ is that it is a topological invariant (in four dimensions), i.e. connected to the Euler characteristic which determines the topology of the manifold. For our purposes, however, what is of interest is that the variation of $\sqrt{-g}\,\mathcal{G}$ with respect to $g^{\mu\nu}$ is zero. This means that we can always add a linear[4] term in $\mathcal{G}$ to the Lagrangian without affecting the equations of motion. In this manner we can replace an eventual $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ modification in favour of terms containing only R^2 and $R^{\mu\nu}R_{\mu\nu}$. As a result, we can write the most general action linear in second-order curvature invariants as[5]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(R + a\,R^2 + b\,R^{\mu\nu}R_{\mu\nu}\right)\,, \tag{3.29}$$

where a, b are two coefficients of suitable dimensions. The theory described by this action is referred to as fourth-order gravity and has been studied extensively in the literature with some interesting results (see [Sch07] for a review).

It is also possible to include in the action derivatives of the mentioned invariants, such as $\Box R$. However the differential order of the field equations is increased as one adds higher derivative terms. More are the derivatives in the action, higher is the order of field equations. In general for every one order increase in the action one gains a two order increase in the equations of motion. So, for example, the R term leads to second-order field equations, the R^2 term to fourth order, $R\Box R$ to sixth order and $R\Box^2 R$ to eighth order. It seems then not convenient to add too many derivatives in the action if we want to keep the field equations at the lowest differential orders.

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[4]Non linear terms in $\mathcal{G}$ are not in general total divergences.

[5]Actually this is the most general second-order action only in the metric formalism. In the Palatini approach the Ricci tensor $R_{\mu\nu}$ is not the only secon-rank tensor we can form contracting the Riemann tensor. The $K_\mu{}^\beta := g^{\nu\alpha}R_{\mu\nu\alpha}{}^\beta$ contraction is in fact not equivalent to $R_{\mu\nu} := R_{\mu\alpha\nu}{}^\alpha$ since now the connection does not satisfy the metric condition (2.5). In principle the most general Palatini second-order lagrangian should also contain terms as $K_\mu{}^\nu K_\nu{}^\mu$ or $R_{\mu\nu}K^{\mu\nu}$.

In what follows we will not discuss modification to the Einstein-Hilbert action given by other invariants beyond the curvature invariants R. This means that the only new terms we will face with will be just functions of R. However, we will in general allow the lagrangian to be an arbitrary function $f(R)$ of the curvature scalar. The corresponding class of theories has been called $f(R)$ gravity for obvious reasons.

The drastic decision we make has two main explanations. First of all there is enough simplicity: actions based only on $f(R)$ modification are sufficiently general to encapsulate some of the basic features of higher-order gravity, but at the same time they are simple enough to be easy handle. This is why $f(R)$ theories make excellent candidates for toy-theories, from which we can gain some insight into such gravity modifications. Second, there are serious reason to believe that $f(R)$ theories are unique among higher-order gravity theories, in the sense that they seem to be the only ones able to avoid the long known and fatal Ostrograndski instability [Woo07].

3.3 $f(R)$ Theories in Metric Approach

In this section we present the metric formulation of $f(R)$ gravitational theories. First we will write the general $f(R)$ action and derive equations of motion. Then we will show how metric $f(R)$ gravity can be recasted in the form of a Brans-Dicke theory via some conformal transformations. The issues outlined in the present section, and much more details, can be found in general reviews as [DT10] or [SF10].

Action and Field Equations

In $f(R)$ theories of gravity the Einstein-Hilbert action (2.1) is modified considering an arbitrary function f of the curvature scalar R:

$$S := \frac{1}{2}\int d^4x\sqrt{-g}f(R)\,. \tag{3.30}$$

Taking into account also the matter Lagrangian, the general action we must work with reads

$$S_{f(R)} := \int d^4x\sqrt{-g}\left[\frac{1}{2}f(R) + \mathcal{L}_M(g_{\mu\nu},\Psi)\right]\,, \tag{3.31}$$

where as usual Ψ denotes all the matter fields collectively.

Varying action (3.31) with respect to the metric $g^{\mu\nu}$ provides the following gravitational field equations[6]

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = T_{\mu\nu}\,, \tag{3.32}$$

[6]Some issues regarding surface terms will be discussed in section 3.5.

where $T_{\mu\nu}$ is the matter energy-momentum tensor (2.29) and where we have defined[7]

$$F(R) := \frac{\partial f(R)}{\partial R} \,. \tag{3.33}$$

In order to recover GR we just have to set $f(R) = R$, in which case $F(R) = 1$ and eq. (3.32) reduces to the canonical Einstein field equations (2.30).

The trace of eq. (3.32) is

$$F(R)R - 2f(R) + 3\Box F(R) = T \,. \tag{3.34}$$

This equation relates R with T differentially and not algebraically as in GR, where $R = -T$, or in Scalar-Tensor gravity, eq. (3.8). This is an indication that the field equations of $f(R)$ theories will admit a larger variety of solution than Einstein's theory. Without going into details, let us just mention that $T = 0$ no longer implies that $R = 0$, or even constant. Moreover, as we will see, the appearence of the term $\Box F$ implies there is a propagating scalar degree of freedom more than in GR.

Finally we note that is possible to write the gravitational field equation (3.32) in the canonical form

$$G_{\mu\nu} = T_{\mu\nu} + T^{(f)}_{\mu\nu} \tag{3.35}$$

defining the effective energy-momentum tensor depending by $f(R)$ as

$$T^{(f)}_{\mu\nu} := \frac{1}{2} g_{\mu\nu} \left[f(R) - R \right] + \nabla_\mu \nabla_\nu F(R) - g_{\mu\nu} \Box F(R) + \left[1 - F(R) \right] R_{\mu\nu} \,, \tag{3.36}$$

which clearly vanishes for $f(R) = R$.

Equivalence with Brans-Dicke Theories

Now we show how metric $f(R)$ gravity can be regarded as equivalent to some particular Brans-Dicke theory using the tool of field manipulations, which consists generally in redefining, introducing or trasforming fields appearing in the action. We state that two theories are dynamically equivalent if, under a suitable redefinition of the involved fields, one can make their field equations coincide[8].

With this definition two dynamically equivalent theories can be regarded just as different representations of the same theory. The issue of distinguishing between truly different theories and different representations of the same theory is an intricate one, expecially when conformal transformations are used in order to redefine the fields. It goes beyond the scope of the present work to analyze in depth this issue and we refer to [SFL08] and references

[7] Mind to not confuse $F(R)$ given here with $F(\phi)$ defined in section 3.1. The choice of this little abuse of notation will become clear in section 3.5.

[8] This stetement holds only within a classical perspective.

therein for a detailed discussion. Here we simply mention that, given that they are handled carefully, field redefinitions and different representations of the same theory constitute useful tools for understanding gravitational theories.

Having pointed out these premises, we present how the identification between $f(R)$ gravity and Brans-Dicke theories takes place. We start from the $f(R)$ action (3.31) and introduce a new scalar field σ. Then we write the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[f(\sigma) + f'(\sigma)(R-\sigma)\right] + S_M(g_{\mu\nu}, \Psi)\,, \tag{3.37}$$

where f is the same function as in action (3.31) and a prime denotes differentiation with respect to σ[9]. Variation with respect to σ leads to the equation

$$f''(\sigma)(R-\sigma) = 0\,, \tag{3.38}$$

which, imposing the condition $f''(\sigma) \neq 0$, gives $R = \sigma$. Substituting this result into action (3.37) reproduces action (3.31) showing that the two theories are indeed dynamically equivalent[10].

Redefining the field σ by $\phi := f'(\sigma)$, action (3.37) becomes

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi R - V(\phi)\right] + S_M(g_{\mu\nu}, \Psi)\,, \tag{3.39}$$

where we have setted

$$V(\phi) := \frac{1}{2}\left[\sigma(\phi)\phi - f(\sigma(\phi))\right]\,. \tag{3.40}$$

Action (3.39) is nothing but a Brans-Dicke action (3.24) with non-vanishing potential and BD parameter $\omega_0 = 0$. What we have just demonstrated is the dynamical equivalence between metric $f(R)$ gravity and the $\omega_0 = 0$ Brans-Dicke theory.

Before ending the section, we write down the equations of motion for this theory. Varying action (3.39) with respect to $g_{\mu\nu}$ and ϕ provides respectively

$$G_{\mu\nu} = \frac{1}{\phi}\left[T_{\mu\nu} - g_{\mu\nu}V(\phi) + \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right]\,; \tag{3.41}$$

$$R = 2\,V'(\phi)\,; \tag{3.42}$$

In order to replace R into eq. (3.42) we can take the trace of eq. (3.41). We get

$$3\Box\phi + 4V(\phi) - 2\phi\,V'(\phi) = T\,, \tag{3.43}$$

[9]In this (and the following) section we use a prime to denote differentiation with respect to the only showed argument. For example, $y'(x)$ means the derivative of the function y with respect to the variable x.

[10]The same analysis could be performed at the level of the equations of motion.

which determines the dynamics of ϕ for given matter sources.

We can now clarify why metric $f(R)$ gravity has a dynamical scalar degree of freedom more than GR. In the Brans-Dicke representation the issue is more transparent since this new degree of freedom is naturally represented by the scalar field ϕ. The absence of a kinetic term for ϕ in the action (3.39) or in eq. (3.42) should not mislead us to think that this degree of freedom does not carry dynamics. As we can realize by eq. (3.43), ϕ is dynamically related to the matter source and, therefore, it is a dymanical degree of freedom. Finally, we notice that eq. (3.42) constrains the dynamics of ϕ implying that $f(R)$ gravity, in its $\omega_0 = 0$ Brans-Dicke form, differs from the general (free) Brans-Dicke theory and then constitutes a special case of such theory.

3.4 $f(R)$ Theories in Palatini Approach

As in Scalar-Tensor theories, the Palatini formulation of $f(R)$ gravity is more complicated than the corresponding metric case. In this section we will see how the Palatini variation of $f(R)$ theories brings to consistent gravitational field equations and how, in analogy with the metric formulation, we can recast it in the form of a specific Scalar-Tensor theory. Again, we refer to the extensive reviews [DT10, SF10] for more details.

Field Equations

In Palatini $f(R)$ gravity f is function of $\hat{R}$ rather than of the canonical curvature scalar R. The action then (including matter) reads

$$S_{f(R)} := \int d^4x\sqrt{-g}\left[\frac{1}{2}f(\hat{R}) + \mathcal{L}_M(g_{\mu\nu}, \Psi)\right] . \tag{3.44}$$

Independent variation with respect to $g^{\mu\nu}$ and $\hat{\Gamma}^{\lambda}_{\mu\nu}$ gives respectively

$$F(\hat{R})\hat{R}_{(\mu\nu)} - \frac{1}{2}f(\hat{R})g_{\mu\nu} = T_{\mu\nu}\,; \tag{3.45}$$

$$\hat{\nabla}_\lambda\left(\sqrt{-g}F(\hat{R})g^{\mu\nu}\right) - \hat{\nabla}_\sigma\left(\sqrt{-g}F(\hat{R})g^{\sigma(\mu}\right)\delta^{\nu)}_\lambda = 0\,; \tag{3.46}$$

where $\hat{\nabla}_\mu$ is the covariant derivative with respect to $\hat{\Gamma}^{\lambda}_{\mu\nu}$ and

$$F(\hat{R}) := \frac{\partial f(\hat{R})}{\partial \hat{R}}\,. \tag{3.47}$$

Imposing $f(\hat{R}) = \hat{R}$ equations (3.45) and (3.46) reduce to the corresponding canonical ones (2.33) and (2.40), respectively. According to the arguments of section 2.3 we can then find in this case the usual Einstein field equations (2.30), which means that GR can be consinstently recovered.

If we take the trace of eq. (3.45) we get

$$F(\hat{R})\hat{R} - 2f(\hat{R}) = T\,. \tag{3.48}$$

Putting a part for a moment the difference between R and $\hat{R}$, we can qualitatively compare eq. (3.48) with eq. (3.34). We notice that in the Palatini case the dynamical term $\Box F$ does not appear. This is an early indication that, unlike the metric case, Palatini $f(R)$ gravity does not contain any further scalar degree of freedom than GR. As it happened for the metric formulation, this issue, which will be of central importance in the results of chapter 6, will be clearer within the Scalar-Tensor representation of $f(R)$ gravity.

Again, in analogy with eq. (2.40), eq. (3.46) implies

$$\hat{\nabla}_\lambda \left(\sqrt{-g}F(\hat{R})g^{\mu\nu}\right) = 0\,. \tag{3.49}$$

Following the same procedure we made for Palatini Scalar-Tensor gravity, we define a new metric $h_{\mu\nu}$ conformally connected to $g_{\mu\nu}$ by

$$h_{\mu\nu} := F(\hat{R})g_{\mu\nu}\,. \tag{3.50}$$

Eq. (3.49) becomes then

$$\hat{\nabla}_\mu \left(\sqrt{-h}\,h^{\alpha\beta}\right) = 0\,, \tag{3.51}$$

which implies the metric condition for $\hat{\Gamma}^\lambda_{\mu\nu}$ with the metric $h_{\mu\nu}$, that is

$$\hat{\nabla}_\mu h_{\alpha\beta} = 0\,. \tag{3.52}$$

The independent connection $\hat{\Gamma}^\lambda_{\mu\nu}$ becomes then the Levi-Civita connection with respect to $h_{\mu\nu}$ and can be written as

$$\begin{aligned}\hat{\Gamma}^\lambda_{\mu\nu} &= \frac{1}{2}h^{\lambda\sigma}(h_{\nu\sigma,\mu} + h_{\mu\sigma,\nu} + h_{\mu\nu,\sigma}) \\ &= \Gamma^\lambda_{\mu\nu} + \frac{1}{2F}\left[2\delta^\lambda_{(\mu}\partial_{\nu)}F + g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma F\right]\,.\end{aligned} \tag{3.53}$$

We can now use this relation in order to replace $\hat{\Gamma}^\lambda_{\mu\nu}$ into the gravitational field equations (3.45). The result is

$$\begin{aligned}&F\,R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + \frac{3}{2F}(\nabla_\mu F)(\nabla_\nu F) - \nabla_\mu\nabla_\nu F \\ &\quad -\frac{1}{2}g_{\mu\nu}\left[(1-\frac{3}{F})\nabla_\sigma\nabla^\sigma F + \frac{3}{2F}(\nabla_\sigma F)(\nabla^\sigma F)\right] = T_{\mu\nu}\,,\end{aligned} \tag{3.54}$$

which can again be recasted in the canonical form $G_{\mu\nu} = T_{\mu\nu} + T^{(f)}_{\mu\nu}$ defining the effective energy-momentum tensor

$$T^{(f)}_{\mu\nu} = (1-F)R_{\mu\nu} - \frac{3}{2F}(\nabla_\mu F)(\nabla_\nu F) + \nabla_\mu\nabla_\nu F$$
$$+\frac{1}{2}g_{\mu\nu}\left[(f-R) + (1-\frac{3}{F})\nabla_\sigma\nabla^\sigma F + \frac{3}{2F}(\nabla_\sigma F)(\nabla^\sigma F)\right] . \quad (3.55)$$

Finally, comparing this result with the corresponding result in the metric case (3.36) we see that also in the $f(R)$ theories of gravity the Palatini and metric formulations correspond to different theories rather than two different representations of the same theory. This is evident at the level of the equations of motion where we find more modifying terms considering the Palatini variation rather than the metric one.

Equivalence with Brans-Dicke Theories

In the same way metric $f(R)$ gravity is dynamically equivalent to a $\omega_0 = 0$ Brans-Dicke theory, the Palatini formulation of $f(R)$ theories has its corresponding equivalent among Brans-Dicke theories. The demonstration is similar to the one we presented in the metric case. We introduce the scalar field σ and write the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[f(\sigma) + f'(\sigma)(\hat{R}-\sigma)\right] + S_M(g_{\mu\nu}, \Psi) , \quad (3.56)$$

which is dynamically equivalent to (3.44) (again only if $f''(\sigma) \neq 0$). Replacing the scalar field σ with $\phi := f'(\sigma)$ yields to the the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi\hat{R} - V(\phi)\right] + S_M(g_{\mu\nu}, \Psi) , \quad (3.57)$$

where the potential $V(\phi)$ is defined exactly as in (3.40).

Even though this action is formally the same of (3.39), it does not correspond to a Brans-Dicke theory with $\omega_0 = 0$ since $\hat{R}$ is clearly not equivalent to R. However, exactly as we have done with the gravitational field equations, we can replace $\hat{\Gamma}^\lambda_{\mu\nu}$ in favour of $\Gamma^\lambda_{\mu\nu}$ into action (3.57). In this manner we have

$$\hat{R} = R + \frac{3}{2\phi^2}(\partial\phi)^2 + \frac{3}{\phi}\Box\phi , \quad (3.58)$$

and we can rewrite action (3.57) (modulo surface terms) as

$$S := \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi R + \frac{3}{4\phi}(\partial\phi)^2 - V(\phi)\right] + S_M(g_{\mu\nu}, \Psi) . \quad (3.59)$$

This is exactly the action of a Brans-Dicke theory with BD parameter $\omega_0 = -3/2$, which means we have succeeded in recasting Palatini $f(R)$ gravity in the form of a Brans-Dicke theory. However, unlike the metric case, now the BD parameter does not vanish and, as a consequence, the kinetic term appears explicitly into action (3.59).

The corresponding field equations of action (3.59) are obtained varying independently with respect to $g_{\mu\nu}$ and the scalar ϕ:

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{\phi}\left[T_{\mu\nu} - g_{\mu\nu}V(\phi) + \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right] \\ &\qquad -\frac{3}{2\phi^2}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\right]\,; \end{aligned} \tag{3.60}$$

$$\Box\phi = \frac{\phi}{3}\left[R - 2\,V'(\phi)\right] + \frac{1}{2\phi}\nabla^\mu\phi\nabla_\mu\phi\,. \tag{3.61}$$

Comparing these equations with the corresponding ones in the metric case, eq. (3.41) and eq. (3.42), we immediatly note that in the Palatini case several more terms appear. This is due to the non vanishing of the BD parameter ω_0, which assumes the value $\omega_0 = -3/2$.

In particular, eq. (3.61) is completely different from eq. (3.42) and seems to provide some dynamics for ϕ, which is further confirmed by the kinetic term now appearing in action (3.59). However, once again this is misleading. We can convince ourself taking the trace of eq. (3.60) in order to replace R in eq. (3.61). The result is

$$4V(\phi) - 2\phi\,V'(\phi) = T\,. \tag{3.62}$$

From this equations it is clear that ϕ is in this case algebraically related to the matter source and, therefore, it carries no dynamics of its own. It is indeed possible to eliminate ϕ from the equations of motion combinig equations (3.60) and (3.62), the resulting field equations correspond to eq. (3.54). Thus, the Palatini formulation of $f(R)$ gravity has the same dynamical degrees of freedom as GR. In this case, the scalar degree of freedom arising in the metric case, and represented by ϕ, is absent. Using the Palatini formalism the scalar field ϕ, appearing in the equivalent Brans-Dicke formulation of the theory, plays the only role of an auxiliary field since it can be eliminated from the final form of the field equations. This crucial difference between the metric and Palatini formulation of $f(R)$ gravity will be of relevant importance in the physical observables we will compute in chapters 5 and 6.

3.5 Generalized Gravity

In this section we present a completely general modification of the Einstein-Hilbert action which includes as subclasses both Scalar-Tensor and $f(R)$ theories. In this case, the deviation from GR is given by an arbitrary function depending on both the curvature scalar R and a scalar field ϕ. We will name this theory with the appellative of *Generalized Gravity* in order to denote the general modifying character of the theory. There are no specific reviews throughout the literature interely dedicated to this subject. However, the basic elements can be found in some works analyzing applications of the theory to cosmological perturbations. References to them will be given separately for metric and Palatini formulations.

The following analysis is performed more carefully than in the previous sections, though not so pedagogical as in chapter 2, and more technical details are discussed. The reason why we have not done this before is that both Tensor-Scalar and $f(R)$ theories can be considered specific examples within the wider class of generalized gravity. In this sense any results valid for generalized gravity must also hold for the two subclasses of theories. The remarks pointed out here can thus be equally applied to the previous theories.

Metric Formulation

In this section we analyze generalized gravity within the metric formalism. We derive equations of motion and discuss some important remarks. What follows can be found, though in less details, in [Hwa96, CF08].

We write the action of (metric) generalized gravity in the following form

$$S_{GG} = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(R,\phi) - \frac{1}{2}\omega(\phi)(\partial\phi)^2 - V(\phi) + \mathcal{L}_M(g_{\mu\nu},\Psi)\right] , \tag{3.63}$$

where $f(R,\phi)$ is a general function of both the curvature scalar R and the scalar field ϕ, while $\omega(\phi), V(\phi)$ are arbitrary functions of ϕ and $(\partial\phi)^2 := g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. $\mathcal{L}_M$ is the usual matter Lagrangian depending only by $g_{\mu\nu}$ and all the matter fields Ψ. Reductions to Scalar-Tensor or $f(R)$ theories are obtained setting $f(R,\phi) = F(\phi)R$ or $\phi = 0$, respectively. In order to simplify the following equations we define

$$F(R,\phi) := \frac{\partial f(R,\phi)}{\partial R} . \tag{3.64}$$

We can now justify the abuse of notation used in the previous sections. According to definition (3.64), the functions $F(R,\phi)$ reduces to $F(\phi)$ in the Scalar-Tensor gravity case (where $f(R,\phi) = F(\phi)R$) and to $F(R) = f'(R)$ in the $f(R)$ gravity case (where $f(R,\phi) = f(R)$).

The equations of motion are obtained varying independently with respect to $g^{\mu\nu}$ and ϕ. We obtain[11]

$$FR_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \Box F \quad = \quad T^{(\phi)}_{\mu\nu} + T_{\mu\nu}\,; \tag{3.66}$$

$$\Box\phi + \frac{1}{2\omega}\left(\omega_{,\phi} \nabla^\mu \phi \nabla_\mu \phi - 2V_{,\phi} + f_{,\phi}\right) \quad = \quad 0\,; \tag{3.67}$$

where $T_{\mu\nu}$ is the usual matter energy-momentum tensor (2.29) and $_{,\phi}$ means differentiation with respect to ϕ. The scalar field energy-momentum tensor is defined as in (3.3), i.e.

$$T^{(\phi)}_{\mu\nu} := \omega \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu}[\frac{1}{2}\omega(\partial\phi)^2 + V(\phi)]\,. \tag{3.68}$$

Once again the modified Klein-Gordon equation (3.67) turns out to depend on the curvatur scalar R through the modifying function $f(R, \phi)$. We can still substitute R into eq. (3.67) taking a solution of the trace of eq. (3.66), which reads

$$FR - 2f + 3\Box F = T^{(\phi)} + T\,. \tag{3.69}$$

Eq. (3.69) is now a differential equation in R instead of being an algebraic relation connectiong R to the matter (and ϕ) as it happens in Scalar-Tensor gravity (see eq. (3.8)) or in GR. The situation is similar to the one we encountered in $f(R)$ theories (see eq. (3.34)). In general this means that it will be more difficoult to decouple the two field equations and that the gravitational field equations (3.66) will admit a wider class of solution than GR.

Finally we can rewrite the gravitational field equations (3.66) in the canonical form

$$G_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T_{\mu\nu} + T^{(f)}_{\mu\nu}\,, \tag{3.70}$$

defining the effective energy momentum tensor

$$T^{(f)}_{\mu\nu} := \frac{1}{2} g_{\mu\nu}\left(f - R\right) + \nabla_\mu \nabla_\nu F - g_{\mu\nu}\Box F + (1 - F) R_{\mu\nu}\,. \tag{3.71}$$

---

[11] In this variation we get the surface term

$$\int_{\partial V} d^3x \sqrt{|k|} F(R, \phi)\, \delta K\,, \tag{3.65}$$

which, unlike in the variation of the Einstein-Hilbert action (see note 7 at page 16), is not the total variation of a quantity, due to the presence of $F(R, \phi)$. This means that it is not possible to "heal" the action just by subtracting some surface term before making the variation. The way out comes by notice that action (3.63) includes higher-order derivatives of the metric and therefore it is possible to fix more degrees of freedom on the boundary than those of the metric itself. By setting these degrees of freedom and the scalar field to vanish on the boundary, it is then possible to make the surface term going to zero.

We should mention that interpreting $T^{(f)}_{\mu\nu}$ as an energy-momentum tensor could be measleading. Although the covariant conservation of $T^{(f)}_{\mu\nu}$ follows from the covariant conservations of $G_{\mu\nu}$, $T_{\mu\nu}$ and $T^{(\phi)}_{\mu\nu}$, the effective energy density derived from $T^{(F)}_{\mu\nu}$ is not positive-defined and none of the energy conditions holds. However it could turn out useful when one considers applications to cosmology since $T^{(f)}_{\mu\nu}$ can be put into the form of a perfect fluid energy-momentum tensor.

Palatini Formulation

We now repeat the same analysis for generalized gravity in the Palatini formulation. The following presentation is taken, and expanded, from [TC10]. For an alternative reference see [KK06].

We begin rewriting action (3.63) within the Palatini formulation, where the connection is considered completely independent from the metric,

$$S_{GG} = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(\hat{R},\phi) - \frac{1}{2}\omega(\phi)(\partial\phi)^2 - V(\phi) + \mathcal{L}_M(g_{\mu\nu},\Psi)\right]. \tag{3.72}$$

As before, $\hat{R}$ represents the curvature scalar composed with the independent connection $\hat{\Gamma}^\lambda_{\mu\nu}$.

Independent variation with respect to the metric $g^{\mu\nu}$, the connection $\hat{\Gamma}^\lambda_{\mu\nu}$ and the scalar field ϕ gives, respectively,

$$F(\hat{R},\phi)\hat{R}_{\mu\nu} - \frac{1}{2}f(\hat{R},\phi)g_{\mu\nu} = T_{\mu\nu} + T^{(\phi)}_{\mu\nu}\,; \tag{3.73}$$

$$\hat{\nabla}_\mu\left[\sqrt{-g}\,F(\hat{R},\phi)\,g^{\alpha\beta}\right] - \delta^{(\beta}_\mu\hat{\nabla}_\sigma\left[\sqrt{-g}\,F(\hat{R},\phi)\,g^{\alpha)\sigma}\right] = 0\,; \tag{3.74}$$

$$\hat{\nabla}^\mu\hat{\nabla}_\mu\phi + \frac{1}{2\omega}\left[\omega_{,\phi}\partial^\mu\phi\partial_\mu\phi - 2V_{,\phi} + f(\hat{R},\phi)_{,\phi}\right] = 0\,. \tag{3.75}$$

We stress that the covariant derivative $\hat{\nabla}$ is taken with respect to the independent connection $\hat{\Gamma}^\lambda_{\mu\nu}$, whilst the usual Levi-Civita covariat derivative is denoted with ∇.

Once given the form of the function f, we can obtain $\hat{R}$ from the trace of eq. (3.73),

$$F(\hat{R},\phi)\hat{R} - 2f(\hat{R},\phi) = T + T^{(\phi)}\,, \tag{3.76}$$

and replace it into eq. (3.75). In this manner eq. (3.75) will result coupled to gravity only through the metric $g_{\mu\nu}$ and will not depend on the curvature scalar $\hat{R}$. Note that eq. (3.76) relates $\hat{R}$ to the matter sources algebraically and not differentially as it happens within the metric formulation.

We now focus on eq. (3.74). Exactly as we noticed in canonical GR (see eq. (2.40)), this constitutes a system of 40 equations in the 40 variables

$\hat{\nabla}_\mu \left[\sqrt{-g}\, F(\hat{R},\phi)\, g^{\alpha\beta}\right]$ which admits the only solution[12]

$$\hat{\nabla}_\mu \left[\sqrt{-g}\, F(\hat{R},\phi)\, g^{\alpha\beta}\right] = 0\,. \tag{3.77}$$

We now define a new metric $h_{\mu\nu}$ conformally connected to $g_{\mu\nu}$ by

$$h_{\mu\nu} := F(\hat{R},\phi) g_{\mu\nu}\,, \tag{3.78}$$

in terms of which eq. (3.77) reads

$$\hat{\nabla}_\mu \left(\sqrt{-h}\, h^{\alpha\beta}\right) = 0\,. \tag{3.79}$$

This implies[13]

$$\hat{\nabla}_\lambda h_{\mu\nu} = 0\,, \tag{3.80}$$

which is nothing but the metric condition (2.5) for the connection $\hat{\Gamma}^\lambda_{\mu\nu}$ with respect to the metric $h_{\mu\nu}$. We can then write $\hat{\Gamma}^\lambda_{\mu\nu}$ as

$$\hat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} h^{\lambda\sigma}(h_{\nu\sigma,\mu} + h_{\mu\sigma,\nu} + h_{\mu\nu,\sigma})\,, \tag{3.81}$$

which, using the definition of $h_{\mu\nu}$ (3.78), becomes

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2F}\left[2\delta^\lambda_{(\mu}\partial_{\nu)}F + g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma F\right]\,, \tag{3.82}$$

where $\Gamma^\lambda_{\mu\nu}$ is the usual Levi-Civita connection with respect to $g_{\mu\nu}$ (eq. (2.4)). Note that the functions F appearing on the right hand side of eq. (3.82) still depend on $\hat{R}$. However, we can always replace $\hat{R}$ in F taking a solution of eq. (3.76) once the form of $f(\hat{R},\phi)$ is given. We can then replace $\hat{\Gamma}^\lambda_{\mu\nu}$ in favour of $\Gamma^\lambda_{\mu\nu}$ and matter whenever it appears in our equations. In this sense we can say that $\hat{\Gamma}^\lambda_{\mu\nu}$ plays the role of an auxiliary field since in principle it can be eliminated from the final form of the field equations.

We now use eq. (3.82) to replace $\hat{\Gamma}^\lambda_{\mu\nu}$ into eq. (3.73) and eq. (3.75). For this scope, and also because it will turn out useful later on, we need to rewrite the hatted Ricci tensor,

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2F^2}(\nabla_\mu F)(\nabla_\nu F) - \frac{1}{F}\nabla_\mu\nabla_\nu F - \frac{1}{2F}g_{\mu\nu}\nabla_\sigma\nabla^\sigma F\,, \tag{3.83}$$

from which follows the hatted curvature scalar,

$$\hat{R} = R - \frac{3}{F}\nabla_\mu\nabla^\mu F + \frac{3}{2F^2}(\nabla_\mu F)(\nabla^\mu F)\,. \tag{3.84}$$

[12] To find this solution just take the trace and replace the result into eq. (3.74).

[13] To obtain eq. (3.80) from (3.79) we must specify how $\hat{\nabla}_\mu$ acts on tensor densities. This is defined exactly as eq. (2.37) but with g substituted by h. Remind that $h^{\mu\nu}$ is the inverse tensor (matrix) of $h_{\mu\nu}$.

The gravitational field equations can then be written as

$$
\begin{aligned}
&F\,R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + \frac{3}{2F}(\nabla_\mu F)(\nabla_\nu F) - \nabla_\mu\nabla_\nu F \\
&\quad -\frac{1}{2}g_{\mu\nu}\left[(1-\frac{3}{F})\nabla_\sigma\nabla^\sigma F + \frac{3}{2F}(\nabla_\sigma F)(\nabla^\sigma F)\right] = T_{\mu\nu} + T^{(\phi)}_{\mu\nu}\,, \qquad (3.85)
\end{aligned}
$$

which can be putted into the canonical form

$$
G_{\mu\nu} = T_{\mu\nu} + T^{(\phi)}_{\mu\nu} + T^{(f)}_{\mu\nu}\,, \qquad (3.86)
$$

defining

$$
\begin{aligned}
T^{(f)}_{\mu\nu} &= (1-F)R_{\mu\nu} - \frac{3}{2F}(\nabla_\mu F)(\nabla_\nu F) + \nabla_\mu\nabla_\nu F \\
&\quad +\frac{1}{2}g_{\mu\nu}\left[(f-R) + (1-\frac{3}{F})\nabla_\sigma\nabla^\sigma F + \frac{3}{2F}(\nabla_\sigma F)(\nabla^\sigma F)\right]\,. \qquad (3.87)
\end{aligned}
$$

We stress that F is still function of $\hat{R}$ in the equations of motion. However, as said above, once one knows the root of eq. (3.76), F becomes function only of the matter sources. In this sense, looking at eq. (3.86), the theory can be brought to the form of GR with a modified source.

Finally we notice that the Palatini gravitational equations (3.86) are different from the correspondig equations within the metric formulation, eq. (3.70). This means that the two formalisms correspond to physically distinguished theories and not only to two different formulations of the same theory. In fact, as we will see in chapter 6, the two approaches will lead to different values of some inflationary observables.

Chapter 4

Inflation and Cosmological Perturbations

This chapter is a review of inflationary cosmology. For the moment we leave a part any modification of gravity considering only GR as the gravitational theory we need for our purposes. The reader interested in inflationary cosmology within modified theories of gravity can find the subject in the next chapter.

In what follows we will present the elementary features of inflation and discuss the topic of cosmological perturbation theory. First we will provide some elementary elements of general cosmology, which will turn out to be useful for the following issues. Then, we will present the Flatness and Horizon problems, showing how inflationary cosmology can solve them and how we can theoretically achive inflation with the aid of a scalar field. Finally, we will treat cosmological perturbations focusing, at the very end, on their application to inflationary cosmology.

The following review is not intended to be a complete treatment of inflationary standard theory. The arguments exposed are only the ones needed in order to achieve a (self-consistent) presentation of inflationary perturbation theory. This presentation will not comprehend all the possible applications of cosmological perturbation theory to inflationary physics. We will appeal only to the material necessary to derive the scalar and tensor spectral indices, namely the main physical observables of inflationary cosmology.

For more details and further material we refer to the huge amount of reviews one can find in literature. There many textbooks on treating inflationary cosmology, among which we cite [Muk05, Lin05, Dod03, Wei08]. Other useful sources are the extensive review articles on the subject, for example [BTW06, Rio02, LR99]. Some specific references will be also given throughout the text.

4.1 Basic Cosmology

In this section we briefly review the basic elements of modern cosmology being of crucial importance for the topics treated from now on. The presented material can be found in all classical textbooks of cosmology, among which we pick out [Wei72].

Assuming the large scale homogeneity and isotropy of the universe (Cosmological Principle), the standard cosmology is based upon the maximally spatially symmetric Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] , \qquad (4.1)$$

where $a(t)$ is the scale factor of the universe and $K = 0, 1, -1$ is the spatial curvature. The coordinates, r, θ, and φ, are referred to as comoving coordinates: a particle at rest in these coordinates remains at rest, i.e. at constant r, θ, and φ. The FRW metric is sometimes written in terms of the *conformal time* η defined by $dt = a\, d\eta$, i.e.

$$\eta := \int \frac{dt}{a(t)} . \qquad (4.2)$$

In these coordinate system we have

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] , \qquad (4.3)$$

which, in the case $K = 0$, reduces to a metric conformally connected to the Minkowski metric:

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right) , \qquad (4.4)$$

The conformal time will turn out useful later, when we will recall it in order to simplify some calculations.

The equations governing the cosmic behaviour are of course the Einstein field equations[1] (2.30):

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = T_{\mu\nu} , \qquad (4.5)$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor.

The matter inside a homogeneous and isotropic universe can be described, at large scales and with high precision, as a perfect fluid. Its energy-momentum tensor is solely determined by its energy density ρ and isotropic (no shear nor viscosity) pressure P:

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P\, g_{\mu\nu} , \qquad (4.6)$$

[1] In this chapter we do not consider any deviation from GR. See chapters 5 and 6 for cosmology in modified gravity.

with the vector u_μ denoting the four-velocity of an observer comoving with the fluid. In cosmology a key role is played by the equation of state relating the energy density $\rho(t)$ and the pressure $P(t)$,

$$P = w\rho\,, \tag{4.7}$$

where w is the equation of state parameter, which takes the values $w = 0$ and $w = 1/3$ for pressureless matter (dust) and radiation, respectively.

Combining the FRW metric (4.1) with the Einstein field equations (4.5), we obtain the standard cosmological equations dominating the universe evolution. These are the well-known Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{K}{a^2}\,, \tag{4.8}$$

and acceleration (Raychaudhuri) equation

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6}\,. \tag{4.9}$$

$H(t) := \dot{a}/a$ is called Hubble parameter (rate) and can be used to measure the expansion velocity of the universe. Here a dot means differentiation with respect to the time coordinate t. We can also combine eq. (4.8) and eq. (4.9) to obtain an equation for $\dot{H}$,

$$2\dot{H} = -\rho - P + \frac{2K}{a^2}\,. \tag{4.10}$$

Eq. (4.8) and eq. (4.9) (or equivalently eq. (4.10)) determine the time evolution of the function $a(t)$ once the functions $\rho(t)$ and $P(t)$ are known.

Finally, we notice that from the conservation equation $\nabla_\lambda T_{\mu\nu} = 0$ we get the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0\,, \tag{4.11}$$

which, once given the equation of state (4.7), provides the following solution of $\rho(t)$ in terms of $a(t)$:

$$\rho \propto a^{-3(1+w)}\,. \tag{4.12}$$

Hence, in the physical cases of matter ($w = 0$) and radiation ($w = 1/3$) dominated universes we get respectively

$$\rho_m \propto a^{-3} \qquad \text{and} \qquad \rho_r \propto a^{-4}\,. \tag{4.13}$$

These inserted back into eq. (4.8) and eq. (4.9) give the time evolution of the scale factor in the flat ($K = 0$) matter and radiation dominated universes as, respectively,

$$a \propto t^{2/3} \qquad \text{and} \qquad a \propto t^{1/2}\,. \tag{4.14}$$

The scale factor is then an increasing function of the coordinate time t in both universes, but the expansion velocity decreases in time (decelerated expansion), as can be seen computing the respective Hubble parameters

$$H = \frac{2}{3t} \qquad \text{and} \qquad H = \frac{1}{2t}\,. \tag{4.15}$$

The (old) standard Big-Bang cosmological model requires a radiation dominated era followed by a matter dominated era, i.e. an always decelerating universe. As we will see in the next section, this implies serious troubles for early time cosmology.

4.2 Flatness and Horizon Problems

In this section we analyze the main shortcomings of Big-Bang cosmology. In particular we discuss the Flatness and Horizon problems showing why the old standard Big-Bang model cannot address them.

Flatness Problem

The best way to explain the Flatness problem is achieved rewriting the Friedmann equation in terms of the critical energy density. This is defined setting $K = 0$ in eq. (4.8) in which case we have

$$\rho_{crit} := 3H^2\,. \tag{4.16}$$

In the standard cosmological model, the critical energy density determines the amount of energy for which the universe will collapse at some instant in the future ($\rho_{tot} > \rho_{crit}$ with $K > 1$) or will expand eternally ($\rho_{tot} \leq \rho_{crit}$ with $K \leq 1$). The energy density of any matter field in the universe can be expressed as a fraction Ω of ρ_{crit}:

$$\Omega = \frac{\rho}{\rho_{crit}}\,. \tag{4.17}$$

The Friedmann equations (4.8) can then be rewritten as

$$\Omega - 1 = \frac{K}{a^2H^2}\,, \tag{4.18}$$

which clearly shows that if $\rho = \rho_{crit}$ the universe is spatially flat $K = 0$. In a matter or radiation dominated universes we have the solutions (4.14) which inserted in eq. (4.18) give respectively

$$|\Omega - 1| \propto t^{2/3} \qquad \text{or} \qquad |\Omega - 1| \propto t\,. \tag{4.19}$$

This means that Ω tends to evolve away from unity with the expansion of the universe.

The Flatness problem can now be formulated. Recent astronomical observations suggest that our universe is spatially flat within a few percent of accuracy, i.e. we have $\Omega \simeq 1$ today. If we assume the standard (decelerated) Big-Bang cosmology, this result forces Ω to be much closer to unity in the past. For example, if we require $|\Omega - 1| < 0.02$ today, we have $|\Omega - 1| < 10^{-61}$ at Planck time. This appears to be an extreme fine-tuning of initial conditions. Unless initial conditions are chosen very accurately, the universe either collapses too soon or expands too quickly before the structure can be formed. This is exactly what we mean with the term Flatness problem.

Horizon Problem

Consider a comoving wavelength λ and the corresponding physical wavelength $a\lambda$, which at some time is inside the Hubble radius H^{-1} (i.e. $a\lambda \leq H^{-1}$). As we can see from the results (4.14), standard big-bang decelerating cosmology is characterized by the cosmic evolution of $a \propto t^n$ with $0 < n < 1$. In this case the physical wavelength grows as $a\lambda \propto t^n$, whereas the results (4.15) tell us that the Hubble radius evolves as $H^{-1} \propto t$. Therefore the physical wavelength becomes much smaller than the Hubble radius at late times. Conversely any finite comoving scale becomes much larger than the Hubble scale at early times. This means that a causally connected region can only be a small fraction of the Hubble radius. The Horizon problem can then be posed as follows: how is it possible that today we observe an extremely isotropic universe also at scales well beyond the Hubble radius if at those scales the galaxies have never been causally connected throughout the entire hystory of the universe?

The most striking example of the Horizon problem is exposed in the CMB. We need first to introduce the comoving particle horizon defined as

$$d_H(t_0) := \int_{t_*}^{t_0} \frac{dt}{a(t)}\,, \tag{4.20}$$

which is the largest distance that a particle could have traveled from an initial time t_* to today t_0. d_H tells us whether two events has ever been in causal contact, while the causal connected region at time t is given by the particle horizon $D_H(t) := a(t)d_H(t)$. This, for $a(t) \propto t^n$, can be approximated to Hubble radius H^{-1} which is then a good estimation of the causal connected region at time/scale factor $a(t)$. Within the standard cosmological model, we can estimate the ratio of the comoving particle horizon at decoupling $d_H(t_{dec})$ to the particle horizon today $d_H(t_0)$. We find

$$\frac{d_H(t_{dec})}{d_H(t_0)} \simeq \left(\frac{t_{dec}}{t_0}\right)^{1/3} \simeq \left(\frac{10^5}{10^{10}}\right)^{1/3} \simeq 10^{-2}\,. \tag{4.21}$$

This implies that the causally connected regions at last scattering are much smaller than the horizon size today. This appears to be at odds with observations of the CMB which presents the same temperature to high precision (at least to one part in 10^4) in all directions in the sky. Yet there is no way to establish thermal equilibrium if these points were never in causal contact before last scattering.

The problem is even more severe if we consider the size of a causal region at the Planck time t_P. We can estimate that

$$\frac{d_H(t_P)}{d_H(t_0)} \simeq 10^{-26} \,. \tag{4.22}$$

Thus the universe we see today should be made up of about 10^{78} regions which were causally disconnected at the Planck time and yet the distribution of matter was extremely smooth over this whole region. This, again, is what we call horizon problem.

4.3 The Standard Model of Inflation

The simplest way to address the problems outlined in the previous section is to consider an inflationary primordial phase, where the universe undergoes an accalerated expansion. In what follows we show how those problems are solved and how we can theoretically obtain the needed expansion introducing a scalar field minimally coupled to gravity.

The Idea of Inflation

Acceleretating the universe expansion means to require an accelerated scale factor $a(t)$. The idea of inflationary cosmology is then to impose

$$\ddot{a} > 0 \,. \tag{4.23}$$

Considering the acceleration equation (4.9), this condition implies

$$\rho + 3P < 0 \,, \tag{4.24}$$

or, in term of the equation of state parameter,

$$w < -\frac{1}{3} \,. \tag{4.25}$$

This correspond to violating the strong energy condition, or, physically speaking, to consider negative values for the pressure P (assuming ρ still positive). The condition (4.23) essentially means that $\dot{a}$ $(= aH)$ increases during inflation and hence that the comoving Hubble radius $(aH)^{-1}$ decreases in the inflationary phase. It is now easy to see how this idea solves the Flatness and Horizon problems.

Flatness Problem. We again appeal to the Friedmann equation written in the form of eq. (4.18). Since during the inflationary period aH rapidly increases with time, Ω is immediately driven toward unity. Then, after inflation stops, the universe undergoes a radiation and a matter dominated eras as dictated by standard cosmology and $|\Omega - 1|$ begins to increas again. However, as long as the inflationary expansion lasts sufficiently long and drives Ω very close to 1, Ω will remain close to unity even in the present epoch. In this manner, we can explain the observed late time spatial flatness of the universe.

Horizon Problem. The Horizon problem can be solved looking at the comoving particle horizon (4.20). This can be rewritten as

$$d_H = \int_{a_*}^{a_0} \frac{1}{aH} \frac{da}{a} = \int_{a_*}^{a_0} \frac{1}{aH} \, d(\ln a) \,, \tag{4.26}$$

i.e. as the logarithmic integral over the history of the comoving Hubble radius $(aH)^{-1}$. The inflationary trick consists in assuming a sufficiently large Hubble radius at the beginning of inflation (such that all the events in the universe were in causal contact), which then decrises in size during the subsequently inflationary epoch. In this way the current Hubble radius could be smaller than in the past but the comoving particle horizon could be still much larger, i.e.

$$d_H(t_0) \gg \frac{1}{a(t_0)H(t_0)} \,. \tag{4.27}$$

This means that, even though all the galaxies we observe are not causally connected now, they were in causal contact before inflation took place. During this period they had the possibility to thermalize providing, in this way, the present highly isotropic universe. This solves the Horizon problem.

Of course the Hubble radius begins to grow after inflation ends, during the standard cosmological eras. In order to solve the horizon problem, it is required that the following condition is satisfied for the comoving particle horizon:

$$\int_{t_*}^{t_{dec}} \frac{dt}{a(t)} \gg \int_{t_{dec}}^{t_0} \frac{dt}{a(t)} \,, \tag{4.28}$$

i.e. the comoving distance which photons can travel before decoupling needs to be much larger than that after the decoupling. This is achieved when the universe expands at least about e^{62} times during inflation, or, in other words, we need a number of e-foldings at least of

$$N \simeq 62 \,, \tag{4.29}$$

where N is defined by the relation

$$a(t_f) = e^N a(t_i) \,, \tag{4.30}$$

where t_i and t_f are respectively the initial and final times of the inflationary period.

Scalar Field Cosmology

The question now is: how we can realize the inflationary requirement (4.23)? Or equivalently, what kind of matter has an equation of state satisfying condition (4.24)? The answer is: *scalar field.*

We can achieve this scenario introducing a minimally coupled scalar field ϕ into the Einstein-Hilbert action (2.1). This scalar field is called *inflaton.* The full gravitational action we must consider is then

$$S := \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \mathcal{L}_\phi\right) , \tag{4.31}$$

where the scalar field Lagrangian is

$$\mathcal{L}_\phi := -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) , \tag{4.32}$$

with $V(\phi)$ is a generic potential. There are no matter fields during the inflationary stage[2] (except for the inflaton). Varying the $\mathcal{L}_\phi$ term in action (4.31) with respect to $g^{\mu\nu}$ gives the stress-energy tensor

$$T^{(\phi)}_{\mu\nu} := \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] , \tag{4.33}$$

which is nothing but (3.68) with $\omega = 1$.

The main assumption we make on the inflaton field is that it has to be homogeneous, i.e. it is assumed to be function of the time only:

$$\phi(x_\mu) \to \phi(t) . \tag{4.34}$$

In this case, all the spatial derivatives in the energy-momentum tensor (4.33) vanish and $T^{(\phi)}_{\mu\nu}$ can be written in the form of a perfect fluid energy-momentum tensor

$$T^{(\phi)}_{\mu\nu} = (\rho_\phi + P_\phi)\, u_\mu u_\nu + P_\phi\, g_{\mu\nu} , \tag{4.35}$$

defining the scalar field energy density and pressure to be, respectively,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) ; \tag{4.36}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) ; \tag{4.37}$$

[2] All the known particles were created when inflation finished during a phase called Reheating, which will be not discussed in our review.

with an overdot denoting differentiation with respect to the coordinate time t. We can verify this statement computing the 00-component and the spatial trace of $T^{(\phi)}_{\mu\nu}$; we get $T^{(\phi)}_{00} = \rho_\phi$ and $T^{(\phi)i}{}_i = 3P_\phi$, which correspond to the same results we obtain from the perfect fluid energy-momentum tensor (4.35). Consequently, the inflaton equation of state parameter reads

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\,, \tag{4.38}$$

which, in the limit $V(\phi) \gg \dot{\phi}^2$, becomes $w_\phi \simeq -1$ (negative pressure) satisfying the inflationary condition (4.25). In conclusion, we have found that, as long as the potential energy is much greater than the kinetic energy, a homogeneous scalar field (the inflaton) can drive the inflationary expansion.

Inflationary Dynamics

The first models of scalar field inflation[3] used a scalar field trapped in a false vacuum. Since $\dot{\phi}$ is small in the local minima then the scalar field is potential dominated with a constant value of the potential which gives a constant Hubble rate and an exponentially growing scale factor (de Sitter solution). These first, simple models suffered from the problem of how to end inflation [HMS82, GW83]. The only way to stop inflating for ϕ was to (quantum) tunnel to the true vacuum region. However, this is a viable scenario only if the tunneling happens at the same time at every spatial point[4], which clearly represents a violation of causality.

Alternatives scenarios, where the field is falling toward the true vacuum and is not trapped in a local minimum, were soon proposed in [Lin82, AS82]. In order to still inflate, the scalar field has to satisfy some specific conditions, which make the inflaton to *slow-roll* toward the true minimum. In what follows we review the basic elements of slow-roll inflation deriving the conditions the inflaton must obey.

We first need equations of motion from the inflationary action (4.31). Variation with respect to the metric $g^{\mu\nu}$ gives of course the Einstein field equations $G_{\mu\nu} = T^{(\phi)}_{\mu\nu}$, from which, using the FRW metric (4.1), we obtain the Friedmann and acceleration equations: (4.8) and (4.9) (with $\rho, P = \rho_\phi, P_\phi$ now). Hence, with the definitions (4.36) and (4.37), eqs. (4.8) and (4.10) read[5]:

$$3H^2 = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,; \tag{4.39}$$

[3] Among which the very first is [Gut81].

[4] Otherwise the region still in the false vacuum would be inflated away from the expanding bubbles [HMS82, GW83].

[5] Since the observed spatial curvature of the universe is practically zero (Flatness problem), from now on we set $K = 0$.

$$2\dot{H} = -\rho_\phi - P_\phi = -\dot{\phi}^2 \,. \tag{4.40}$$

Similarly, varying action (4.31) with respect to the scalar field ϕ gives the canonical Klein-Gordon equation $\Box\phi - V_{,\phi} = 0$, which using the FRW metric becomes[6]

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \,. \tag{4.41}$$

We now introduce the slow-roll conditions. We assume that the potential $V(\phi)$ is such that

$$\dot{\phi}^2 \ll V(\phi) \qquad \text{and} \qquad \ddot{\phi} \ll H\dot{\phi} \,. \tag{4.42}$$

We have already seen that the first of these two conditions is necessary in order for the inflaton equation of state to satisfy the constraint (4.25). The second one means that we are considering an almost flat potential, at least in the inflationary region, which implies the inflaton is slowly rolling down to the true vacuum. In this manner, the scalar field can remain for sufficiently long time in the inflationary region where the first condition ensures inflation.

Imposing the inflationary conditions (4.42), the field equations (4.39) and (4.41) become respectively

$$3H^2 \simeq V \,; \tag{4.43}$$
$$3H\dot{\phi} \simeq -V_{,\phi} \,. \tag{4.44}$$

It is now useful to define the so-called slow-roll parameters:

$$\epsilon_1 := -\frac{\dot{H}}{H^2} \qquad \text{and} \qquad \epsilon_2 := \frac{\ddot{\phi}}{H\dot{\phi}} \,. \tag{4.45}$$

Using eqs. (4.43) and (4.44) we can relate these parameters to the form of the potential $V(\phi)$. In fact, taking into account eq. (4.40) and eq. (4.43), the square and the time derivative of eq. (4.44) gives, respectively,

$$\epsilon_1 \simeq \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad \text{and} \qquad \epsilon_2 - \epsilon_1 \simeq \frac{V_{,\phi\phi}}{V} \,. \tag{4.46}$$

The slow-roll parameters are of crucial importance in inflationary cosmology since, as we will see later, they can be connected to some physical observables. Measuring these observables will then give us the values of ϵ_1 and ϵ_2, which, through eqs. (4.46), will provide useful information about the structure of the inflationary potential $V(\phi)$.

[6] Alternatively, one can derive this equation from the continuity equation (4.11) using the definitions (4.36) and (4.37).

We can also rewrite the slow-roll conditions (4.42) in terms of the slow-roll parameters (4.45). It is easy to see that the second condition of (4.42) becomes exactly

$$\epsilon_2 \ll 1\,, \tag{4.47}$$

while deriving (in time) the Hubble rate H we find

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2(1-\epsilon_1)\,. \tag{4.48}$$

This equation tells us that, in order to satisfy the main inflationary condition $\ddot{a} > 0$, we must have $\epsilon_1 < 1$, i.e.

$$\text{inflation} \iff \epsilon_1 < 1\,. \tag{4.49}$$

As soon as this condition fails, inflation ends. Commonly, slow-roll inflationary cosmology assumes $\epsilon_1 \ll 1$, in which case we have

$$\text{slow-roll inflation} \iff \epsilon_1, \epsilon_2 \ll 1\,. \tag{4.50}$$

Moreover, thanks to the results (4.46), during inflation the slow-roll parameters ϵ_1 and ϵ_2 can be considered to be approximately constant since the potential $V(\phi)$ is almost flat[7], i.e.

$$\text{slow-roll inflation} \iff \dot{\epsilon}_1, \dot{\epsilon}_2 = 0\,. \tag{4.51}$$

We have then found how the inflationary conditions are expressed in terms of the inflationary parameter ϵ_1, ϵ_2 and how these parameter can be related to the inflaton potential.

Finally, we show how the inflationary potential $V(\phi)$ is also connected to an important observable, namely the number of e-foldings N. Using eqs. (4.43) and (4.44) we have

$$\begin{aligned} N &= \int_{a_i}^{a_f} \frac{da}{a} = \int_{t_i}^{t_f} H\,dt \\ &= \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}}\,d\phi = \int_{\phi_i}^{\phi_f} \frac{3H^2}{3\dot{\phi}H}\,d\phi \\ &\simeq -\int_{\phi_i}^{\phi_f} \frac{V}{V_{,\phi}}\,d\phi\,, \end{aligned} \tag{4.52}$$

where ϕ_i and ϕ_f are the values of the inflaton field at the beginning and end of inflation, respectively.

[7]We can also see this considering that $\dot{\epsilon}_1, \dot{\epsilon}_2 = \mathcal{O}(\epsilon_1^2, \epsilon_2^2)$, which means that condition (4.51) holds at first order in ϵ_1, ϵ_2 (recall that ϵ_1, ϵ_2 are small because of (4.50)).

4.4 Cosmological Perturbations

The predictive power of the inflationary standard model relies on its ability to generate primordial perturbations acting as seeds for the late time cosmological structure formation. Consequently, the theory of relativistic perturbations represents the main tool of modern cosmology, where we find applications throughout the entire history of the universe. The subject is too wide to be treated in depth here, but we will provide the basic elements we need to derive the primordial power spectra, which, as we will see, are the observables related to the slow-roll parameters and thus, thanks to eqs. (4.46), to the shape of the inflaton potential.

In this section we introduce the formalism of cosmological perturbations defining notation, discussing gauge issues and deriving equations governing their evolution. The following section will be devoted to apply these results to the inflationary universe in order to compute power spectra.

Metric Perturbations

First of all we show how general metric perturbations can be split and classified according to their transformation property under spatial rotations. General perturbations of the metric can be written as

$$ds^2 = (g_{\mu\nu} + \delta g_{\mu\nu})\, dx^\mu dx^\mu \,, \tag{4.53}$$

where $g_{\mu\nu}$ is the background metric and $\delta g_{\mu\nu}$ characterizes the perturbations. For small perturbations we require $\delta g_{\mu\nu} \ll g_{\mu\nu}$. According to the Cosmological Principle, the cosmological background metric is assumed to be a general FRW metric (4.1), which we recall here:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] . \tag{4.54}$$

The metric perturbations $\delta g_{\mu\nu}$ can be categorized into three distinct types: scalar, vector and tensor perturbations. This classification is based on the symmetry properties of the homogeneous, isotropic background, which at any given moment of time is obviously invariant with respect to the group of spatial rotations and translations.

The δg_{00} component behaves as a scalar under these rotations and hence we can label it introducing the 3-scalar α:

$$\delta g_{00} = -2\alpha \,. \tag{4.55}$$

The spacetime components δg_{0i} can be decomposed into the sum of the spatial gradient of some scalar β and a vector b_i with zero divergence ($\partial_i b^i = 0$):

$$\delta g_{0i} = -2a(t)\left(\partial_i\beta - b_i\right) . \tag{4.56}$$

In a similar fashion, the components δg_{ij}, which behave as a tensor under spatial rotations, can be written as the sum of the following irreducible pieces:

$$\delta g_{ij} = 2\psi g_{ij} + a^2(t)\left(2\nabla_i\nabla_j\gamma + 2\nabla_{(i}c_{j)} + h_{ij}\right) . \tag{4.57}$$

Here ψ and γ are 3-scalars, c_i is a divergenceless ($\partial_i c^i = 0$) 3-vector and h_{ij} is a transverse ($\partial_i h^i{}_j = 0$), traceless ($h^i{}_i = 0$) and symmetric ($h_{ij} = h_{ji}$) 3-tensor. The general perturbed FRW metric reads then

$$\begin{aligned} ds^2 = &-(1+2\alpha)\,dt^2 - 2a\,(\partial_i\beta - b_i)\,dt\,dx^i \\ &+ \left[(1+2\psi)g_{ij} + a^2\left(2\nabla_i\nabla_j\gamma + 2\nabla_{(i}c_{j)} + h_{ij}\right)\right] dx^i dx^j . \end{aligned} \tag{4.58}$$

which in a spatially flat universe ($g_{ij} = a^2\delta_{ij}$) reduces to

$$\begin{aligned} ds^2 &= -(1+2\alpha)\,dt^2 - 2a\,(\partial_i\beta - b_i)\,dt\,dx^i \\ &\quad + a^2\left(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma + 2\partial_{(i}c_{j)} + h_{ij}\right) dx^i dx^j . \end{aligned} \tag{4.59}$$

We can now classify these perturbations according to their behaviour under spatial rotations. Scalar perturbations are characterized by the four scalar functions α, β, γ, ψ, which are induced by energy density inhomogeneities. These perturbations are important because they exhibit gravitational instability and may lead to the formation of structure in the universe. Vector perturbations are described by the two vectors b_i and c_i and are related to the rotational motions of the (cosmological) fluid. Usually they decay very quickly and are not very interesting from the point of view of cosmology. Tensor perturbations are represented only by h_{ij}. They describe gravitational waves, which are the degrees of freedom of the gravitational field itself. In the linear approximation gravitational waves do not induce any perturbations in the perfect fluid. The main advantage of this decomposition is that at linear order scalar, vector and tensor perturbations are decoupled and can be analyzed separately. This will be very useful later on since during the primordial inflationary phase all the perturbations are small, which means we can analyze them at first order.

Finally, we count the number of independent functions composing $\delta g_{\mu\nu}$. From the scalar perturbations α, β, γ, ψ, we clearly get four independent functions. The vector perturbations b_i and c_i has in total four independent components being both divergence free. The symmetric tensor h_{ij} has to satisfy four constrains, which means it is composed only by two independent components. Adding all these together we arrive at ten independent functions characterizing the perturbation $\delta g_{\mu\nu}$, which amount exactly to the independent components of the background metric $g_{\mu\nu}$.

Gauge Transformations

Let us consider the coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \zeta^\mu(x) \,, \tag{4.60}$$

where ζ^μ are infinitesimally small functions ($\zeta^\mu \ll x^\mu$) of the coordinates x^α. At any given spacetime point, the usual transformation law of the metric gives

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta(x)} \simeq g_{\alpha\beta}(x) + \delta g_{\alpha\beta} - 2 g_{\gamma(\alpha}\partial_{\beta)}\zeta^\gamma \,, \tag{4.61}$$

where we have kept only the terms linear in δg and ζ. In the new coordinates $\tilde{x}$ the metric can also be split into background and perturbation parts,

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = g_{\alpha\beta}(\tilde{x}) + \delta\tilde{g}_{\alpha\beta} \,, \tag{4.62}$$

where $g_{\alpha\beta}(\tilde{x})$ is the FRW metric depending on the new coordinates $\tilde{x}^\mu$. Note that at first order in ζ we have

$$g_{\alpha\beta}(x) \simeq g_{\alpha\beta}(\tilde{x}) - \zeta^\gamma \partial_\gamma g_{\alpha\beta} \,. \tag{4.63}$$

Comparing the expressions in (4.61) and (4.62), and taking into account eq. (4.63), we can infer the following gauge transformation law for the cosmological perturbations:

$$\delta g_{\alpha\beta} \to \delta\tilde{g}_{\alpha\beta} = \delta g_{\alpha\beta} - \zeta^\gamma \partial_\gamma g_{\alpha\beta} - 2 g_{\gamma(\alpha}\partial_{\beta)}\zeta^\gamma \,. \tag{4.64}$$

Since physics has to be invariant under coordinate transformations of the kind (4.60), all the observables depending on the perturbations $\delta g_{\alpha\beta}$ has to be invariant under transformations (4.64). In this sense, the (4.64) plays the same role of gauge transformations we find for istance in electrodynamics.

The transformations (4.60) are more commonly written as

$$\tilde{t} = t + \delta t \,, \qquad \tilde{x}^i = x^i + \delta^{ij}\partial_j \delta x + \delta x^i \,, \tag{4.65}$$

which correspond to

$$\zeta^0 = \delta t \,, \qquad \zeta^i = \delta^{ij}\partial_j \delta x + \delta x^i \,. \tag{4.66}$$

What we have done is to split the spatial coordinate parameters ζ^i into a scalar function δx and a divergence free ($\partial_i \delta x^i = 0$) vector δx^i. In this manner, δt and δx characterize the time slicing and the spatial threading, respectively. Using the definitions (4.66), under the gauge transformations (4.64), scalar perturbations behave as

$$\alpha \;\to\; \tilde{\alpha} = \alpha - \dot{\delta t} \,; \tag{4.67}$$
$$\beta \;\to\; \tilde{\beta} = \beta - a^{-1}\delta t + a\dot{\delta x} \,; \tag{4.68}$$
$$\psi \;\to\; \tilde{\psi} = \psi - H\delta t \,; \tag{4.69}$$
$$\gamma \;\to\; \tilde{\gamma} = \gamma - \delta x \,; \tag{4.70}$$

while vector perturbations behave as

$$b_i \quad \to \quad \tilde{b}_i = b_i - a\delta\dot{x}_i\,; \tag{4.71}$$

$$c_i \quad \to \quad \tilde{c}_i = c_i - \delta x_i\,; \tag{4.72}$$

where as usual an overdot denotes differentiation with respect to the coordinate time t. Tensor perturbations h_{ij} are invariant under gauge transformations,

$$\tilde{h}_{ij} = h_{ij}\,. \tag{4.73}$$

Thus, except for tensor perturbations, all the perturbation functions appearing in the (perturbed) FRW metric (4.58) are not gauge invariant. However, using the transformation laws (4.67)-(4.72), we can always build some gauge invariants combining the perturbation functions in a suitable way. The simplest gauge invariants we can define are the two scalars

$$\Phi_{(\alpha)} \quad := \quad \alpha - \frac{d}{dt}\left[a(\dot{\gamma} + \beta)\right]\,; \tag{4.74}$$

$$\Phi_{(\psi)} \quad := \quad -\psi + aH(\dot{\gamma} + \beta)\,; \tag{4.75}$$

and the shear vector

$$\Phi_i^{(V)} \quad := \quad b_i - a\dot{c}_i\,. \tag{4.76}$$

We notice that only δt and δx contribute to the transformations of scalar perturbations and by choosing them appropriately we can make any two of the four functions α, β, γ, ψ vanish. Similarly, only two of the four independent functions b_i, c_i characterize physical perturbations; the other two reflect the coordinate freedom of imposing δx_i (only two conditions because $\partial_i \delta x^i = 0$). The variables (4.74), (4.75) span the two-dimensional space of physical scalar perturbations, while the observables (4.76) span the two-dimensional space of physical vector perturbations, which describe rotational motion.

The gauge issue plays an important role expecially in the analysis of scalar perturbations, for which several possible gauge choices has been considered throughout the literature being any of them more useful in different contexts. The most common are the *Longitudinal (conformal-Newtonian) gauge* and the *Synchronous gauge*. The first is defined by the condition $\beta = \gamma = 0$, which corresponds to set $\delta t = \delta x = 0$. In this case, from eqs. (4.74) and (4.75), we have $\Phi_{(\alpha)} = \alpha$ and $\Phi_{(\psi)} = -\psi$, which give to the gauge variables a direct interpretation. The second imposes the condition $\alpha = \beta = 0$, in order to have $\delta g_{0\alpha} = 0$ for scalar perturbations. This does not fix the coordinates uniquely; there exists a whole class of synchronous coordinate systems. This residual coordinate freedom leads to the appearance of unphysical gauge modes, which render the interpretation of physical results difficult.

Finally, we discuss matter perturbations, i.e. perturbations of the energy-momentum $T_{\mu\nu}$. These are also gauge dependent: energy density (and pressure) perturbations obey the simple transformation rule

$$\delta T_{00} = \delta\rho \to \delta\tilde{\rho} = \delta\rho - \dot{\rho}\,\delta t\,. \tag{4.77}$$

Moreover, dividing the perturbed 3-momentum $T^0{}_i$ into a scalar δq and a divergence-free vector δq_i parts,

$$\delta T^0{}_i := \partial_i \delta q + \delta q_i\,, \tag{4.78}$$

the scalar quantity δq transforms as

$$\delta q \to \delta\tilde{q} = \delta q + (\rho + P)\delta t\,. \tag{4.79}$$

We can now build another very useful gauge invariant defining

$$\mathcal{R} := \psi + \frac{H}{\rho + P}\delta q\,, \tag{4.80}$$

which is often called *curvature perturbation invariant* since it can be related to the spatial curvature on constant time hypersurfaces[8]. For standard inflation we have $\delta q = -\dot{\phi}\delta\phi$ and the curvature invariant becomes

$$\mathcal{R}_{\delta\phi} := \psi - \frac{H}{\dot{\phi}}\delta\phi\,, \tag{4.81}$$

where we have used definitions (4.36) and (4.37) for the scalar field energy density and pressure, respectively.

Cosmological Perturbation Equations

At first order scalar, vector and tensor perturbations are decoupled. This means we can find equations governing their evolution independently. To derive these equations we must perturb the Einstein field equations:

$$\delta G_{\mu\nu} = \delta T_{\mu\nu}\,. \tag{4.82}$$

We leave all the details of this derivation to the given references since we have not enough space here to go through them in depth. However, all the equations needed for our purpose will be reported and discussed.

First of all we define two quantities (not gauge invariants) which will help us to simplify the following equations:

$$\chi := a(\beta + a\dot{\gamma}) \qquad \text{and} \qquad A := 3(H\alpha - \dot{\psi}) - \frac{\triangle}{a^2}\chi\,, \tag{4.83}$$

[8] Mind not to confuse $\mathcal{R}$ with the curvature (Ricci) scalar R.

where $\triangle$ denotes the 3-space Laplacian[9].

Scalar Perturbations. We can now write down equations for scalar perturbations. Perturbing the $G^0{}_0$, $G^0{}_i$, $G^i{}_j - \frac{1}{3}\delta^i_j G^k{}_k$ and $G^k{}_k - G^0{}_0$ components of the field equations we get, respectively,

$$\frac{3K+\triangle}{a^2}\psi + HA = -\frac{1}{2}\delta\rho\,; \tag{4.84}$$

$$H\alpha - \dot{\psi} + \frac{K}{a^2}\chi = -\frac{1}{2}\delta q\,; \tag{4.85}$$

$$\dot{\chi} + H\chi - \psi - \alpha = a^2\delta\Pi\,; \tag{4.86}$$

$$\dot{A} + 2HA + \left(3H + \frac{\triangle}{a^2}\right)\alpha = \frac{1}{2}\left(\delta\rho + \delta P\right)\,; \tag{4.87}$$

where $\delta\Pi$ is the perturbation of the scalar part of the anisotropic stress given by $\Pi_{ij} = \left(\partial_i\partial_j - \frac{\triangle}{3}\delta_{ij}\right)\Pi$. Once the matter distribution is known, these four equations provide the evolution of scalar metric perturbations α, β, γ, ψ (or equivalently α, χ, A, ψ).

Equation (4.86) can be written in terms of gauge metric perturbations $\Phi_{(\alpha)}$ and $\Phi_{(\psi)}$, defined in Eqs. (4.74) and (4.75), as the constraint

$$\Phi_{(\psi)} - \Phi_{(\alpha)} = a^2\delta\Pi\,. \tag{4.88}$$

Thus, in absence of anisotropic stresses, as in the case of a perfect (cosmological) fluid, we have $\Phi_{(\psi)} = \Phi_{(\alpha)}$. The space of physical scalar perturbations becomes then one-dimensional and in order to characterize their evolution we only need to find one equation for one of the gauge invariants constructed previously. As we will see, for our purpuses, this role will be played by the curvature invariant $\mathcal{R}$, defined in eq. (4.80).

In the standard model of inflation the energy-momentum tensor $T_{\mu\nu}$ becomes the scalar field energy-momentum tensor $T^{(\phi)}_{\mu\nu}$ given in eq. (4.33) (or in eq. (4.35)). Moreover, the anisotropic tensor vanishes ($\Pi_{\mu\nu} = 0$) and the universe can be taken spatially flat ($K = 0$). We can then rewrite eqs. (4.84)-(4.87) as

$$\frac{\triangle}{a^2}\psi + HA = -\frac{1}{2}\left(V_{,\phi}\delta\phi + \dot{\phi}\dot{\delta\phi} - \dot{\phi}^2\alpha\right)\,; \tag{4.89}$$

$$H\alpha - \dot{\psi} = \frac{1}{2}\dot{\phi}\delta\phi\,; \tag{4.90}$$

[9]Note that in the Longitudinal gauge $\chi = 0$.

$$\dot{\chi} + H\chi - \psi - \alpha = 0\,; \tag{4.91}$$

$$\dot{A} + 2HA + \left(3H + \frac{\triangle}{a^2}\right)\alpha = V_{,\phi}\delta\phi + 2\dot{\phi}\dot{\delta\phi} - 2\dot{\phi}^2\alpha\,; \tag{4.92}$$

where of course $\delta\phi$ is the perturbation of the inflaton field ϕ. These equations will be of central interest in the next section, where we will derive the inflationary spectral indices.

Vector Perturbations. The vector perturbation part of the $G^0{}_i$ components of the (perturbed) Einstein field equations gives the following evolution equation for vector perturbations:

$$\frac{\triangle + 2K}{2a^2}\Phi_i^{(V)} = \delta q_i\,. \tag{4.93}$$

For a single scalar field energy-momentum tensor $\delta q_i = 0$ and then, in the standard inflationary model ($K = 0$), we have

$$\triangle\Phi_i^{(V)} = 0\,, \tag{4.94}$$

which implies immediately[10] $\Phi_i^{(V)} = 0$. This means that within the standard model of inflation primordial vector perturbations are not produced at all. We can thus ignore their contribution in the analysis of next section.

Tensor Perturbations. What we need is an evolution equation for the tensor perturbation h_{ij}. This is given by the spatial perturbation part of the Einstein field equations and is written as

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{2K - \triangle}{a^2}h_{ij} = 0\,. \tag{4.95}$$

Tensor perturbations describe gravitational waves which can be generally decomposed into two polarization states h_+ and $h_\times$. Any gravitational wave can thus be expanded on a basis formed by these two polarization states:

$$h_{ij} = h(t)\, e_{ij}^{(+,\times)}\,. \tag{4.96}$$

Thus, eq. (4.95) becomes a single evolution equation for the gravitational wave amplitude $h(t)$,

$$\ddot{h} + 3H\dot{h} + \frac{2K - \triangle}{a^2}h = 0\,. \tag{4.97}$$

This is the same as the wave equation for a massless scalar field in an unperturbed FRW metric. Being a second order differential equation any tensor perturbations generated during inflation propagates into the subsequent cosmological eras. As we are going to see, primordial gravitational waves can thus bring some measurable, though small, late time physical effects.

[10] In spatial Fourier space eq. (4.94) reads $k^2\Phi_i^{(V)} = 0$.

4.5 Perturbations Generated During Inflation

Although there are several applications of cosmological pertubation theory, we focus on inflation. In this section we discuss primordial scalar and tensor perturbations[11] deriving spectral indices and the tensor-to-scalar ratio. These are the observable which we can directly related to the shape of the inflaton potential.

Scalar Perturbations

First of all we choose a specific gauge to work in. The most convenient gauge choice for our aim is the so-called *Uniform-field gauge*: $\delta\phi = 0$. Within this gauge the curvature invariant $\mathcal{R}$, as given in eq. (4.81), coincides with the scalar perturbation ψ,

$$\mathcal{R}|_{\delta\phi=0} = \psi \,. \tag{4.98}$$

In the Uniform-field gauge, ψ is thus gauge invariant and can be replaced wherever appears with $\mathcal{R}$. If we succeed in deriving an equation governing $\mathcal{R}$ we find the physical evolution of scalar perturbations, which will be automatically independent from the gauge choice.

In order to simplify the following analysis from now on we will work in spatial Fourier space where $\triangle = -k^2$, with k denoting the comoving wavenumber. Substituting ψ with $\mathcal{R}$ and taking $\delta\phi = 0$, eqs. (4.89)-(4.92) become

$$-\frac{k^2}{a^2}\mathcal{R} + HA = \frac{1}{2}\dot{\phi}^2\alpha\,; \tag{4.99}$$

$$H\alpha - \dot{\mathcal{R}} = 0\,; \tag{4.100}$$

$$\dot{\chi} + H\chi - \mathcal{R} - \alpha = 0\,; \tag{4.101}$$

$$\dot{A} + 2HA + \left(3H - \frac{k^2}{a^2}\right)\alpha = -2\dot{\phi}^2\alpha\,. \tag{4.102}$$

From eq. (4.100) we have immediately

$$\alpha = \frac{\dot{\mathcal{R}}}{H}\,, \tag{4.103}$$

which plugged into eq. (4.99) yields

$$A = \frac{1}{H}\left(\frac{k^2}{a^2}\mathcal{R} + \frac{\dot{\phi}^2}{2H}\dot{\mathcal{R}}\right)\,. \tag{4.104}$$

[11]Vector perturbations are not generated during the inflationary phase (see previous section).

Inserting eqs. (4.103) and (4.104) into eq. (4.102), and taking in account eq (4.40), we get the following evolution equation for the curvature invariant $\mathcal{R}$:

$$\ddot{\mathcal{R}} + \left(3H + \frac{\dot{Q}}{Q}\right)\dot{\mathcal{R}} + \frac{k^2}{a^2}\mathcal{R} = 0\,, \tag{4.105}$$

where we have defined[12]

$$Q := \frac{\dot{\phi}^2}{H^2}\,. \tag{4.106}$$

Introducing the Mukhanov-Sasaki variables $z_s := a\sqrt{Q}$ and $u_s := z_s\mathcal{R}$ [MFB92, Sas86], eq. (4.105) becomes

$$u_s'' + \left(k^2 - \frac{z_s''}{z_s}\right)u_s = 0\,, \tag{4.107}$$

where from now on a prime denotes differentiation with respect to the conformal time η defined in (4.2).

We rewrite eq. (4.107) in terms of slow-roll parameters ϵ_1, ϵ_2, defined in eqs. (4.45), considering slow-roll inflation where $\dot{\epsilon}_1 = \dot{\epsilon}_2 = 0$. If the parameter ϵ_1 is constant, it follows that[13]

$$\eta = -\frac{1}{aH(1-\epsilon_1)}\,. \tag{4.108}$$

We then find [SL93]

$$\frac{z_s''}{z_s} = \frac{\nu_s^2 - 1/4}{\eta^2}\,, \tag{4.109}$$

with

$$\nu_s^2 := \frac{1}{4} + \frac{(1+\epsilon_1+\epsilon_2)(2+\epsilon_2)}{(1-\epsilon_1)^2}\,. \tag{4.110}$$

The solution of eq. (4.107) can now be expressed as a linear combination of Hankel functions of the first $H_\nu^{(1)}$ and second $H_\nu^{(2)}$ kind

$$u_s = \frac{1}{2}\sqrt{\pi|\eta|}e^{i\pi(1+2\nu_s)/4}\left[C_1 H_{\nu_s}^{(1)}(k|\eta|) + C_2 H_{\nu_s}^{(2)}(k|\eta|)\right]\,, \tag{4.111}$$

where C_1, C_2 are constants of integration and ν_s represents the Hankel function index.

[12] Note that, from eq. (4.40), we have also $Q = 2\epsilon_1$.

[13] We have

$$\eta = \int \frac{dt}{da} = \int \frac{da}{a^2H} = -\frac{1}{aH} + \int \frac{\epsilon_1}{a^2H}da\,,$$

where we have used $d(aH) = Hda + adH$. If ϵ_1 is constant and $\epsilon_1 < 1$ (inflationary conditions) we get

$$\eta = -\frac{1}{aH} + \epsilon_1\int\frac{da}{a^2H} = -\frac{1}{aH}\sum_{n=0}^{\infty}\epsilon_1^n = -\frac{1}{aH}\frac{1}{1-\epsilon_1}\,.$$

In the asymptotic past ($k\eta \to -\infty$) the solution to eq. (4.107) is given by

$$u_s \to \frac{1}{\sqrt{2k}} e^{-ik\eta} \qquad (k\eta \to -\infty)\,, \tag{4.112}$$

corresponding to a (quantum) vacuum state in flat space time field theory well inside the (comoving) Hubble radius $(aH)^{-1}$. This fixes the coefficients to be $C_1 = 1$ and $C_2 = 0$, meaning that the general solution of eq. (4.107) we must consider is

$$u_s = \frac{1}{2}\sqrt{\pi|\eta|} e^{i\pi(1+2\nu_s)/4} H^{(1)}_{\nu_s}(k|\eta|)\,. \tag{4.113}$$

On the other side, in the super-horizon limit $k\eta \to 0$, the solution (4.113) becomes

$$u_s \to e^{\frac{i\pi}{4}(2\nu_s - 1)}\, 2^{\nu_s - \frac{3}{2}} \frac{1}{\sqrt{2k}} \frac{\Gamma(\nu_s)}{\Gamma(3/2)} (-k\eta)^{-\nu_s + \frac{1}{2}} \qquad (k\eta \to 0)\,, \tag{4.114}$$

where we have used the asymptotic properties of the Hankel functions. Here Γ is the Euler gamma function[14], for which it holds $\Gamma(3/2) = \sqrt{\pi}/2$.

In slow-roll inflation, at zeroth order in ϵ_1 and ϵ_2, we have $z_s''/z_s \simeq (aH)^2$. For the modes deep inside the Hubble radius ($k \gg aH$, or $|k\eta| \gg 1$) u_s satisfies the standard equation of a canonical field in Minkowski spacetime: $u_s'' + k^2 u_s \simeq 0$. During inflation, after the Hubble radius crossing ($k = aH$), the effect of the gravitational term z_s''/z_s becomes important. In the super-Hubble limit ($k \ll aH$, or $|k\eta| \ll 1$) the last term on the left hand side of eq. (4.105) can be neglected, giving the solution

$$\mathcal{R} = C_3 + C_4 \int \frac{dt}{a^3 Q}\,, \tag{4.115}$$

where C_3, C_4 are integration constants. The second term can be identified as a decaying mode, which rapidly decays during inflation. Hence the curvature perturbation approaches a constant value C_3 after the Hubble radius crossing, which means that all the super-Hubble modes freezes after inflation.

We can now define the power spectrum of curvature perturbations as

$$\mathcal{P}_s := \frac{k^3}{2\pi^2}|\mathcal{R}|^2\,. \tag{4.116}$$

In the super-horizon limit, we can compute $\mathcal{P}_s$ recalling solution (4.114):

$$\mathcal{P}_s \simeq \frac{1}{Q}\left((1-\epsilon_1)\frac{\Gamma(\nu_s)}{\Gamma(3/2)}\frac{H}{2\pi}\right)^2 \left(\frac{|k\eta|}{2}\right)^{3-2\nu_s}\,. \tag{4.117}$$

[14] Not to be confused with the (Levi-Civita) connection $\Gamma^\lambda_{\mu\nu}$.

Since the curvature perturbations are frozen after the Hubble radius crossing, the spectrum (4.117) can be evaluated at $k = aH$. We then define the *scalar spectral index* n_s as

$$n_s - 1 := \left.\frac{d\ln\mathcal{P}_s}{d\ln k}\right|_{k=aH}, \tag{4.118}$$

which, thanks to eq. (4.117), becomes simply

$$n_s - 1 = 3 - 2\nu_s\,. \tag{4.119}$$

Since during inflation we have $\epsilon_1, \epsilon_2 \ll 1$, at first order in the ϵ's the scalar spectral index becomes

$$n_s - 1 \simeq -4\epsilon_1 - 2\epsilon_2\,. \tag{4.120}$$

This result tells us that the spectrum is close to scale invariance: $n_s \simeq 1$ (at zeroth order). A rapid estimation of the scalar power spectrum is then given by

$$\mathcal{P}_s{}^{\frac{1}{2}} \simeq \left.\frac{H^2}{2\pi\dot{\phi}}\right|_{k=aH}. \tag{4.121}$$

The result (4.120) is one of the most important in inflationary cosmology since n_s is a physical observable and can be measured with nowadays experiments. Recently, many of such experiments has been performed and, at the moment, the best value we have is [WMAP09]

$$n_s = 0.960 \pm 0.013\,, \tag{4.122}$$

which clearly confirms the theoretical result given in (4.120). As already mentioned, the importance of measuring n_s relies on the fact that it is connected, through the slow-roll parameter ϵ_1 and ϵ_2 (see eqs. (4.46)), to the shape of the inflaton potential $V(\phi)$. Thus, knowing the values of the slow-roll parameters means to gain some breakthrough into fundamental physics, or at least to achieve some piece of information about the origin of the universe. The problem is that, in order to obtain both ϵ_1 and ϵ_2, the measure of n_s alone does not suffice. We need another independent result, which could be derived analyzing tensor inflationary perturbations.

Tensor Perturbations

The analysis we have just done for scalar perturbations can be repeated for tensor perturbations. As we have seen, also tensor perturbations are generated during the inflationary phase. We remind that the gravitational wave perturbations h_{ij}, and thus also its amplitude h, are gauge invariant. Consequently, determining the evolution of h means to characterize the physical behaviour of tensor perturbations.

In fourier space ($\triangle = -k^2$) and in a spatially flat universe ($K = 0$), the equation governig gravitational waves, eq. (4.97), reads

$$\ddot{h} + 3H\dot{h} + \frac{k^2}{a^2}h = 0\,. \tag{4.123}$$

We notice that this is equal to equation (4.105), except for the additional term in Q which is now missing. We can still define the Mukhanov-Sasaki variable $u_t := ah$ in order to rewrite eq. (4.123) as

$$u_t'' + \left(k^2 - \frac{a''}{a}\right)u_t = 0\,. \tag{4.124}$$

The process can now proceed exactly as before. We express a''/a in terms of slow-roll parameters,

$$\frac{a''}{a} = \frac{\nu_t^2 - 1/4}{\eta^2}\,, \tag{4.125}$$

with

$$\nu_t^2 := \frac{1}{4} + \frac{2-\epsilon_1}{(1-\epsilon_1)^2}\,. \tag{4.126}$$

The solution of eq. (4.124) we consider is

$$u_s = \frac{1}{2}\sqrt{\pi|\eta|}e^{i\pi(1+2\nu_t)/4}H_{\nu_t}^{(1)}(k|\eta|)\,, \tag{4.127}$$

which is normalized taking the short wavelenght limit ($k\eta \to -\infty$).

The spectrum of tensor perturbations $\mathcal{P}_t$ is defined similarly to the scalar one $\mathcal{P}_s$. The only difference comes from considering all the polarization states of the gravitational wave, which, in brief, corresponds to an additional factor of 8 in the definition of $\mathcal{P}_t$:

$$\mathcal{P}_t := 8 \times \frac{k^3}{2\pi^2}|h|^2\,. \tag{4.128}$$

Again, taking the super-horizon limit ($k\eta \to 0$) of solution (4.127), we find

$$\mathcal{P}_t \simeq 8\left((1-\epsilon_1)\frac{\Gamma(\nu_t)}{\Gamma(3/2)}\frac{H}{2\pi}\right)^2\left(\frac{|k\eta|}{2}\right)^{3-2\nu_t} \tag{4.129}$$

In analogy with the scalar analysis, we define the *tensor spectral index* n_t as

$$n_t := \left.\frac{d\ln\mathcal{P}_t}{d\ln k}\right|_{k=aH}\,, \tag{4.130}$$

which, using eq. (4.129), becomes

$$n_s = 3 - 2\nu_t\,. \tag{4.131}$$

Substituting the definition of ν_t, eq. (4.126), at first order in the slow-roll parameters we find

$$n_t \simeq -2\epsilon_1 \,. \tag{4.132}$$

For tensor perturbations the spectral index is close to zero since $\epsilon_1 \ll 1$. Finally, we can obtain a rapid estimation of the tensor perturbation spectrum with

$$\mathcal{P}_t^{\frac{1}{2}} \simeq \left.\frac{H^2}{2\pi}\right|_{k=aH} . \tag{4.133}$$

Finally, we introduce the *tensor-to-scalar ratio* r defined as the ratio between the tensor and scalar power spectra,

$$r := \frac{\mathcal{P}_t}{\mathcal{P}_s} \,. \tag{4.134}$$

Although it is an observable, r is still related to n_s and n_t. The usefulness of r relies on the fact that it is easier to measure than n_t. Instead of measuring directly n_s and n_t we could measure n_s and r, which would probably represent an advantage. With the estimations (4.121) and (4.133), it is possible to evaluate r in terms of the slow-roll parameters,

$$r \simeq 8Q \simeq 16\epsilon_1 \,, \tag{4.135}$$

where in the last step we have taken only linear terms in ϵ_1 and ϵ_2. Eq. (4.135) introduces a consistency relation for r and n_t,

$$r = -8n_t \,, \tag{4.136}$$

which underlines the connection between the two quantities.

The tensor spectral index n_t (or the tensor-to-scalar ratio r) is a physical observable which can, in principle, be measured independently from n_s. Then, the result (4.132) is exaclty the one we need in order to completely determine the values of ϵ_1 and ϵ_2. Once we measure both n_s and n_t, with the reqiured accuracy, the form of the inflaton potential will begin to be unveiled. Unfortunately, neither n_t nor r have yet been measured, because of several difficoulties in detecting the extremely feeble gravitational waves[15]. Some experiments, which, hopefully, will be able to detect the small effects these primordial tensor perturbations cause in the CMB physics, are scheduled for the next future. Thanks to them, we will probably be able to understand something more about inflation and our universe in general.

[15] At the moment, we can only rely on an observational upper bound of $r < 0.22$ [WMAP09].

Chapter 5

Inflationary Perturbations in Modified Gravity: Metric Formulation

The present chapter, together with the last one, represents the main part of this work. Here we study inflationary cosmological perturbations within the metric formulation of Generalized gravity, whilst in the following chapter we will face with the Palatini approach. The aim is to derive scalar and tensor spectral indices and to compare the obtained results in different gravitational theories such as $f(R)$ and Scalar-Tensor Gravity.

First, we introduce the (modified) background cosmological equations in order to outline the differences from the canonical Friedmann (4.8) and acceleration (4.9) equations. Then we present the cosmological perturbation equations and repeat the analysis performed in chapter 4 for both scalar and tensor perturbations. Finally, we discuss the obtained results showing how they reduce in both $f(R)$ and Scalar-Tensor theories. As a particular example, we analyze the Starobinsky model of inflation [Sta80], which can be recasted in the $f(R)$ class of theories.

The style of the chapter is kept at a more technical level in comparison with the previous ones. Accordingly, the text lacks of some (rather long) derivations of important equations, but references to the works where they are presented are always given in place.

The following material can be found in the works studying cosmological perturbations in Generalized Gravity, among which the ones made by Hwang (and Noh) represent, without any doubt, the majority [Hwa90a, Hwa90b, Hwa91, Hwa96, HN96, Hwa97, HN01, HN02, NH01, HN05]. The subject is also summarized in some review articles such as [DT10].

5.1 Background Cosmological Equations

In this section we present the background (cosmological) equations for the theories denoted in section 3.5 with the name of (metric) Generalized Gravity. For the sake of simplicity, we report the corresponding action, eq. (3.63),

$$S_{GG} = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(R,\phi) - \frac{1}{2}\omega(\phi)(\partial\phi)^2 - V(\phi) + \mathcal{L}_M(g_{\mu\nu},\Psi)\right] . \quad (5.1)$$

and field equations, eqs. (3.66) and (3.67),

$$FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = T^{(\phi)}_{\mu\nu} + T_{\mu\nu} ; \quad (5.2)$$

$$\Box\phi + \frac{1}{2\omega}\left(\omega_{,\phi}\nabla^\mu\phi\nabla_\mu\phi - 2V_{,\phi} + f_{,\phi}\right) = 0 ; \quad (5.3)$$

obtained with an independent variation of action (5.1) with respect to $g^{\mu\nu}$ and ϕ, respectively. All the required notation can be found in chapter 3, we just stress that the function F is defined by

$$F(R) := \frac{\partial f(R,\phi)}{\partial R} . \quad (5.4)$$

The aim here is to indroduce the (modified) equations corresponding to the canonical Friedmann equation (4.8) and the acceleration equation (4.9) (or equivalently eq. (4.10)). We assume the universe is still described by the FRW metric[1] (4.1),

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] , \quad (5.5)$$

which will be substituted into the equations of motion (5.2) and (5.3). The two energy-momentum tensor $T_{\mu\nu}$ and $T^{(\phi)}_{\mu\nu}$ can be recasted into the form of a perfect fluid energy-momentum tensor, as we saw in the previous chapter. The scalar field ϕ is still assumed to be homogeneous, so we have

$$\rho_\phi = \frac{1}{2}\omega\dot{\phi}^2 + V(\phi) ; \quad (5.6)$$

$$P_\phi = \frac{1}{2}\omega\dot{\phi}^2 - V(\phi) ; \quad (5.7)$$

as its energy density and pressure, respectively[2]. Once again, an overdot denotes differentiation with respect to the coordinate time t.

[1]This assumption is completely independent of the theory of gravity we choose since it relies only on treating the universe as homogeneous and isotropic.

[2]Note that now ω has not been set to one.

Inserting metric (5.5) in the field equations (5.2) and (5.3) we obtain the following modified cosmological equations (see for example [HN96, NH01, DT10]):

$$3FH^2 + \frac{K}{a^2} = \frac{1}{2}(RF - f) - 3H\dot{F} + \frac{1}{2}\omega\dot{\phi}^2 + V + \rho_M\,; \tag{5.8}$$

$$-2F\dot{H} - \frac{K}{a^2} = \ddot{F} - H\dot{F} + \omega\dot{\phi}^2 + \rho_M + P_M\,; \tag{5.9}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega}\left(\omega_{,\phi}\dot{\phi}^2 + 2V_{,\phi} - f_{,\phi}\right) = 0\,; \tag{5.10}$$

which correspond to the modifications of eqs. (4.8), (4.10) and (4.41). Choosing $f = R$, $\omega = 1$ and $K = 0$ these coincides with the standard inflationary equations. Confronting eqs. (5.8) and (5.9) with eqs. (4.8) and (4.10), it is easy to see that the modifying function $f(R)$ leads to several new terms, which render the cosmological equations completely different from the canonical ones.

5.2 Cosmological Perturbations

Cosmological perturbations are defined in Generalized Gravity exactly as in standard GR. All the arguments discussed in section 4.4 still hold here. The only exception are the cosmological perturbation equations which, relying on the cosmolgical background equations, will now take another form. In this section we present the cosmological perturbation equations which will be used in the following inflationary perturbations analysis.

Again, we divide metric perturbations according to their transformation properties under spatial rotations: we have four scalar functions α, β, ψ, γ, two (divergenceless) vector b_i, c_i, and the (gravitational wave) tensor perturbations h_{ij}. The quantities $\Phi_{(\alpha)}$, $\Phi_{(\psi)}$ and $\Phi_i^{(V)}$ are still gauge invariant, as well as

$$\mathcal{R}_{\delta\phi} := \psi - \frac{H}{\dot{\phi}}\delta\phi\,. \tag{5.11}$$

We can also construct another useful gauge invariant from the perturbation of the function F, namely δF. Under gauge transformations, this quantity behaves as

$$\delta F \to \tilde{\delta F} = \delta F - \dot{F}\delta t\,. \tag{5.12}$$

If we then define the scalar

$$\mathcal{R}_{\delta F} := \psi - \frac{H}{\dot{F}}\delta F\,, \tag{5.13}$$

this will be invariant under cosmological gauge transformations. $\mathcal{R}_{\delta F}$ will turn out useful soon, when we will decide the best gauge chioce we need for inflationary cosmological perturbations in Generalized Gravity.

We can now present the (modified) cosmological perturbation equations. Once again, the quantities

$$\chi := a(\beta + a\dot{\gamma}) \qquad \text{and} \qquad A := 3(H\alpha - \dot{\psi}) - \frac{\triangle}{a^2}\chi \,, \tag{5.14}$$

will help us to simplify the following equations.

Scalar Perturbations. Perturbing the field equations (5.2) we find the following equations governing scalar type perturbations [HN96, HN02, HN05, DT10]

$$\begin{aligned} \frac{\triangle + 3K}{a^2}\psi + HA = -\frac{1}{2F}\Bigg[&\left(3H^2 + 3\dot{H} + \frac{\triangle}{a^2}\right)\delta F - 3H\dot{\delta F} \\ &+\frac{1}{2}\left(\omega_{,\phi}\dot{\phi}^2 + 2V_{,\phi} - f_{,\phi}\right)\delta\phi + \omega\dot{\phi}\dot{\delta\phi} \\ &+\left(3H\dot{F} - \omega\dot{\phi}^2\right)\alpha + \dot{F}A + \delta\rho_M\Bigg]\,; \end{aligned} \tag{5.15}$$

$$H\alpha - \dot{\psi} + \frac{3K}{a^2}\chi = \frac{1}{2F}\left[\omega\dot{\phi}\delta\phi + \dot{\delta F} - H\delta F - \dot{F}\alpha - \delta q_M\right]; \tag{5.16}$$

$$\dot{\chi} + H\chi - \alpha - \psi = \frac{1}{F}\left(\delta F - \dot{F}\chi\right)\,; \tag{5.17}$$

$$\begin{aligned} \dot{A} + 2HA + \left(3\dot{H} + \frac{\triangle}{a^2}\right)\alpha = \frac{1}{2F}\Bigg[&3\ddot{\delta F} + 3H\dot{\delta F} - \left(6H^2 + \frac{\triangle + 6K}{a^2}\right)\delta F \\ &+4\omega\dot{\phi}\dot{\delta\phi} + \left(2\omega_{,\phi}\dot{\phi}^2 - 2V_{,\phi} + f_{,\phi}\right)\delta\phi \\ &-3\dot{F}\dot{\alpha} - \left(4\omega\dot{\phi}^2 + 3H\dot{F} + 6\ddot{F}\right)\alpha \\ &-\dot{F}A + \delta\rho_M + \delta P_M\Bigg]\,. \end{aligned} \tag{5.18}$$

These equations are obtained by perturbing the $G^0{}_0$, $G^0{}_i$, $G^i{}_j - \frac{1}{3}\delta^i_j G^k{}_k$ and $G^k{}_k - G^0{}_0$ components of the field equations, respectively. Anisotropic perturbations $\delta\Pi_{\mu\nu}$ has been assumed to vanish.

The new modified perturbation equations (5.15)-(5.18) appear to be rather more complicated than the canonical ones (4.84)-(4.87). Several new terms, depending on the form of $f(R,\phi)$ through its derivative F, seem to preclude any easy manipulation. However, as we will see in the next section, choosing the right gauge will considerably simplify these equations.

Vector Perturbations. Perturbing the $G^0{}_i$ components of the field equations (5.2) we obtain the following equation for vector cosmological perturbations [HN96, HN02]

$$\frac{\triangle + 2K}{2a^2}\Phi_i^{(V)} = \frac{\delta q_i^M}{F}\,, \tag{5.19}$$

where $\Phi_i^{(V)}$ is defined in (4.76). The only deviation from the canonical correspondig equation (4.94) is represented by a modulation of the matter source coupling through the modifying function F. Anyway, if we still assume no matter fields, except the scalar field ϕ, we have $\delta q_i^M = 0$, which means that the right hand side of equation (5.19) vanishes. The situation becomes exactly the same we had in the standard model of inflation: eq. (5.19) implies $\Phi_i^{(V)} = 0$, i.e. no vector perturbations are generated during the inflationary phase. Consequently, in the next section analysis we will not discuss vector inflationary perturbations.

Tensor Perturbations. In absence of anisotropic stresses ($\Pi_{\mu\nu} = 0$), the perturbation of the $G^i{}_j$ components of the field equations (5.2) provides the following equation for the gravitational wave [HN96, HN02]

$$\ddot{h}_{ij} + \left(3H + \frac{\dot{F}}{F}\right)\dot{h}_{ij} + \frac{2K - \triangle}{a^2}h_{ij} = 0\,. \tag{5.20}$$

Expanding the gravitational wave h_{ij} into the polarization state basis as in (4.96), we can transform eq. (5.20) in an equation for the gravitational wave amplitude h:

$$\ddot{h} + \left(3H + \frac{\dot{F}}{F}\right)\dot{h} + \frac{2K - \triangle}{a^2}h = 0\,. \tag{5.21}$$

Confronting this equation with its canonical corresponding one, eq. (4.97), we notice that there is only one difference appearing in an additional damping term modulated by the form of the function F. In the forthcoming section we will use eq. (5.21) to derive the tensor spectral index in Generalized Gravity.

5.3 Perturbations from Inflation

In this section we derive scalar and tensor spectral indices for (metric) Generalized Gravity. The analysis is performed at a completely general level leaving the discussions of the results we can obtain in specific inflationary model for the next sections.

First of all, we need to choose a gauge, which, again, it is of interest only for scalar perturbations. In the standard model of inflation we chose the Uniform-field gauge, $\delta\phi = 0$. A more convenient choice in modified gravity could be the *Uniform-F gauge*: $\delta F = 0$, depending on the inflationary model we are studying. We will either select the Uniform-field gauge or the Uniform-F gauge depending on which theory we are working with.

The Uniform-F gauge is the more useful to work with when we are restricted to $f(R)$ theories of gravity. In this class of theories all the contributions of the scalar field ϕ (and of its perturbation $\delta\phi$) are absent. From the definitions (5.11) and (5.13) we see immediately that setting $\delta F = \delta\phi = 0$ gives $\mathcal{R} = \mathcal{R}_{\delta F} = \mathcal{R}_{\delta\phi} = \psi$. This means that ψ becomes a gauge invariant and can be substituted with $\mathcal{R}$ whenever it appears. Then, combining equations (5.15)-(5.18) could lead to an equation governing its evolution.

The Uniform-field gauge is instead used in Scalar-Tensor theories of gravity. In this class of theories F depends only on ϕ, i.e. we have $f(R,\phi) = F(\phi)R$, and we get $\delta F = F_{,\phi}\delta\phi$. Choosing $\delta\phi = 0$ immediately gives $\delta F = 0$. These are the same conditions we had in the $f(R)$ theories with the Uniform-F gauge. All the results obtained setting $\delta F = \delta\phi = 0$ will then hold for both $f(R)$ and Scalar-Tensor theories.

Unfortunately, though it includes these two important theories, this analysis does not hold for more general theories, where both a scalar field and a non-linear coupling of R appear. Setting $\delta\phi = 0$ does not imply $\delta F = 0$ since, in that case, we have $\delta F = F_{,\phi}\delta\phi + F_{,R}\delta R$. Another choice of gauge is probably more appropriate in this more general case but we will restrict ourselves to the pure $f(R)$ or $F(\phi)$ case inasmuch as several interesting result are obtained within these theories.

Scalar Perturbations

Accordingly to the previous discussion, in what follows we assume $\delta F = \delta\phi = 0$ and replace ψ with $\mathcal{R}$. Moreover, since all the matter fields vanish during inflation, we set $T_{\mu\nu} = 0$. Then, in a flat universe ($K = 0$) and in spatial Fourier space ($\triangle = -k^2$), eqs. (5.15)-(5.18) read

$$-\frac{k^2}{a^2}\mathcal{R} + HA = -\frac{1}{2F}\left[\left(3H\dot{F} - \omega\dot{\phi}^2\right)\alpha + \dot{F}A\right]; \quad (5.22)$$

$$H\alpha - \dot{\mathcal{R}} = \frac{\dot{F}}{2F}\alpha; \quad (5.23)$$

$$\dot{\chi} + H\chi - \alpha - \mathcal{R} = -\frac{\dot{F}}{F}\chi; \quad (5.24)$$

$$\dot{A} + 2HA + \left(3\dot{H} - \frac{k^2}{a^2}\right)\alpha = -\frac{1}{2F}\left[\left(4\omega\dot{\phi}^2 + 3H\dot{F} + 6\ddot{F}\right)\alpha + 3\dot{F}\dot{\alpha} + \dot{F}A\right]. \quad (5.25)$$

From eq. (5.23) we get immediately

$$\alpha = \frac{\dot{\mathcal{R}}}{Z}\,, \tag{5.26}$$

where, for the sake of simplicity, we have defined the function

$$Z := H + \frac{\dot{F}}{2F}\,. \tag{5.27}$$

Plugging eq. (5.26) into eq. (5.22), we obtain

$$A = -\frac{1}{Z}\left(\frac{3H\dot{F} - \omega\dot{\phi}^2}{2FZ}\dot{\mathcal{R}} - \frac{k^2}{a^2}\mathcal{R}\right)\,. \tag{5.28}$$

Then, substituting eqs. (5.26) and (5.28) into eq. (5.25), and recalling the background equation (5.9), we obtain a second-order differential equation for the curvature perturbation $\mathcal{R}$:

$$\ddot{\mathcal{R}} + \left(3H + \frac{\dot{Q}_m}{Q_m}\right)\dot{\mathcal{R}} + \frac{k^2}{a^2}\mathcal{R} = 0\,, \tag{5.29}$$

where we have defined

$$Q_m := \frac{1}{Z^2}\left(\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}\right)\,. \tag{5.30}$$

We notice that eq. (5.29) has the same form of its corresponding one in the standard model of inflation, eq. (4.105). The only difference appears in the definition of the function Q_m, which is clearly different from Q, defined in eq. (4.106). We also point out the presence of the last term appearing in brackets of eq. (5.30), which depends solely on the modifying function F. This term is of crucial importance for the forthcoming results. In the Palatini formulation, as we will see in the next chapter, it does not appear, implying drastic consequences for the physical observables of the theory.

Starting from eq. (5.29) we can now repeat the procedure performed in the previous chapter. We introduce suitable Mukhanov-Sasaki variable $z_s := a\sqrt{Q_m}$ and $u_s := z_s\mathcal{R}$, in terms of which eq. (5.29) becomes

$$u_s'' + \left(k^2 - \frac{z_s''}{z_s}\right)u_s = 0\,, \tag{5.31}$$

where a prime denotes again differentiation with respect to the conformal time η defined in eq. (4.2). At this point, we introduce the following variables [HN96, Hwa97, HN01, DT10]

$$\epsilon_1 := -\frac{\dot{H}}{H^2}\,, \qquad \epsilon_2 := \frac{\ddot{\phi}}{H\dot{\phi}}\,, \qquad \epsilon_3 := \frac{\dot{F}}{2HF}\,, \qquad \epsilon_4 := \frac{\dot{E}}{2HE}\,, \tag{5.32}$$

where the function E is defined as

$$E := F\left(\omega + \frac{3\dot{F}^2}{2F\dot{\phi}^2}\right). \tag{5.33}$$

The first two variables in (5.32) are nothing but the slow-roll parameters defined in (4.45). The remaining two depends on the modifying function F and in the canonical case, where $F = \omega = 1$, vanish. In terms of parameters (5.32) and function (5.33) the quantity Q_m reads

$$Q_m = \frac{\dot{\phi}^2}{H^2}\frac{E}{F(1+\epsilon_3)^2}. \tag{5.34}$$

Usually, in Generalized Gravity the slow-roll conditions (4.50) and (4.51) are extended to all the ϵ's defined in (5.32). In what follows we then assume

$$\epsilon_i \ll 1 \quad \text{and} \quad \dot{\epsilon}_i = 0 \quad \text{with} \quad i = 1,2,3,4. \tag{5.35}$$

With the conditions $\dot{\epsilon}_i = 0$ and using eq. (4.108), in eq. (5.31) we have [HN96, DT10]

$$\frac{z_s''}{z_s} = \frac{\nu_s^2 - 1/4}{\eta^2}, \tag{5.36}$$

with

$$\nu_s^2 := \frac{1}{4} + \frac{(1+\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)(2+\epsilon_2-\epsilon_3+\epsilon_4)}{(1-\epsilon_1)^2}. \tag{5.37}$$

Note the new terms depending on ϵ_3 and ϵ_4 which did not appear in (4.110). Exactly as in section 4.5, the solution of eq. (5.31) can thus be written as linear combination of Hankel functions,

$$u_s = \frac{1}{2}\sqrt{\pi|\eta|}e^{i\pi(1+2\nu_s)/4}\left[C_1 H_{\nu_s}^{(1)}(k|\eta|) + C_2 H_{\nu_s}^{(2)}(k|\eta|)\right], \tag{5.38}$$

with C_1 and C_2 constant of integration. Again, taking the asymptotic past (short wavelenght) normalization ($C_1 = 1$, $C_2 = 0$), the solution we must consider is

$$u_s = \frac{1}{2}\sqrt{\pi|\eta|}e^{i\pi(1+2\nu_s)/4}H_{\nu_s}^{(1)}(k|\eta|). \tag{5.39}$$

The power spectrum of curvature perturbations is defined in (4.116), and reads

$$\mathcal{P}_s := \frac{k^3}{2\pi^2}|\mathcal{R}|^2. \tag{5.40}$$

In the super-horizon limit ($k\eta \to 0$), using the solution (5.39), we obtain

$$\mathcal{P}_s \simeq \frac{1}{Q_m}\left((1-\epsilon_1)\frac{\Gamma(\nu_s)}{\Gamma(3/2)}\frac{H}{2\pi}\right)^2\left(\frac{|k\eta|}{2}\right)^{3-2\nu_s}. \tag{5.41}$$

We can thus compute the scalar spectral index, defined in eq. (4.118), at horizon crossing:

$$n_s - 1 := \left.\frac{d\ln \mathcal{P}_s}{d\ln k}\right|_{k=aH} = 3 - 2\nu_s\,, \tag{5.42}$$

with ν_s given in eq. (5.37). Since $\epsilon_i \ll 1$ for all $i = 1, 2, 3, 4$, we can evaluate n_s at first order in the ϵ's. The result is

$$n_s - 1 \simeq -4\epsilon_1 - 2\epsilon_2 + 2\epsilon_3 - 2\epsilon_4\,. \tag{5.43}$$

Finally, from eq. (5.41) the scalar power spectrum can be estimated, at zeroth order in the ϵ's, as

$$\mathcal{P}_s \simeq \frac{1}{Q_m}\left(\frac{H}{2\pi}\right)^2\,. \tag{5.44}$$

We can compare the result in (5.43) with the one obtained in canonical inflation (4.120), which can be recovered setting $F = \omega = 1$ in (5.43). Since $\epsilon_i \ll 1$, n_s is again close to scale invariance, $n_s \simeq 1$. However, new terms, depending on the modifying function $f(R, \phi)$ through ϵ_3 and ϵ_4, now appear. These could in principle imply some deviation from the value obtained within the standard model of inflation (4.120). However, since this value has been already measured, the result given by any specific (modified gravity) model of inflation should match the number in (4.122), i.e.

$$n_s = 0.960 \pm 0.013\,. \tag{5.45}$$

We can thus use the experimental value (5.45) in order to constrain the inflationary modified models. Any model which does not reproduce the number in (5.45) happens to be ruled out by experimental data.

Tensor Perturbations

In a flat universe and in spatial Fourier space, eq. (5.21) reads

$$\ddot{h} + \left(3H + \frac{\dot{F}}{F}\right)\dot{h} + \frac{k^2}{a^2}h = 0\,. \tag{5.46}$$

This is similar to eq. (5.29) of scalar perturbations, a part from the difference of the factor F instead of Q_m. We can still define suitable Mukhanov-Sasaki variables as $z_t = a\sqrt{F}$ and $u_t = z_t h$, in terms of which eq. (5.46) becomes

$$u_t'' + \left(k^2 - \frac{z_t''}{z_t}\right)u_t = 0\,. \tag{5.47}$$

The analysis is similar to the one performed for scalar perturbations. Assuming again $\epsilon_i \ll 1$, we obtain [HN01, DT10]

$$\frac{z_t''}{z_t} = \frac{\nu_t^2 - 1/4}{\eta^2}\,, \tag{5.48}$$

with

$$\nu_t^2 := \frac{1}{4} + \frac{(1+\epsilon_1)(2-\epsilon_1+\epsilon_3)}{(1-\epsilon_1)^2}\,. \tag{5.49}$$

The solution of eq. (5.47) is then

$$u_s = \frac{1}{2}\sqrt{\pi|\eta|}e^{i\pi(1+2\nu_s)/4}H^{(1)}_{\nu_s}(k|\eta|)\,, \tag{5.50}$$

and, in the super-horizon limit, the tensor power spectrum, defined in (4.128), takes the form

$$\mathcal{P}_t := 8 \times \frac{k^3}{2\pi^2}|h|^2 \simeq \frac{8}{F}\left((1-\epsilon_1)\frac{\Gamma(\nu_t)}{\Gamma(3/2)}\frac{H}{2\pi}\right)^2\left(\frac{|k\eta|}{2}\right)^{3-2\nu_t}\,. \tag{5.51}$$

The tensor spectral index, defined in (4.130), follows

$$n_t := \left.\frac{d\ln\mathcal{P}_t}{d\ln k}\right|_{k=aH} = 3-2\nu_t\,, \tag{5.52}$$

with ν_t given in eq. (5.49). At first order in the ϵ's, we have

$$n_t \simeq -2\epsilon_1 - 2\epsilon_3\,, \tag{5.53}$$

which generalizes the result obtained in (4.132). Again, from eq. (5.51) follows the rapid estimation of the power spectrum,

$$\mathcal{P}_t \simeq \frac{8}{F}\left(\frac{H}{2\pi}\right)^2\,. \tag{5.54}$$

Similarly to scalar perturbations, the result (5.53) is modified by the function $f(R,\phi)$ through the parameter ϵ_3. Anyway, since n_t has not been measured yet, tensor perturbations do not help to constrain the modified gravity models of inflation. As we will see, different models predict different result for n_t, which consequently, once measured, will provide an useful tool to discriminate between various inflationary theories.

Finally, we compute the tensor-to-scalar ratio defined in (4.134). Using the rapid estimations (5.44) and (5.54), we find

$$r := \frac{\mathcal{P}_s}{\mathcal{P}_t} \simeq \frac{8Q_m}{F} \simeq 16\,(\epsilon_1+\epsilon_3)\,, \tag{5.55}$$

where in the last step we have used eq. (5.9) and kept only terms to first order in the ϵ's. Although the result (5.55) clearly differs from (4.135), the consistency relation (4.136) is preserved: we still have $r = -8n_t$ (at least to first order). Thus, exactly as it happens in canonical inflation, the two observables n_t and r are directly related.

5.4 Example 1: $f(R)$ Theories

At this point, we discuss how the previous general results reduce in some specific subclasses of theories. In this section we analyze (metric) $f(R)$ gravity, which was introduced in section 3.3.

The very first model of inflation is exactly an $f(R)$ modification of the Einstein-Hilbert action. Even before Guth proposed its inflationary model based on a scalar field trapped in a false vacuum [Gut81], Starobinsky had found that a R^2 gravitational lagrangian could indeed drive inflation [Sta80]. The form of $f(R)$ for this model is

$$f(R) = R + \frac{R^2}{6M^2}\,, \tag{5.56}$$

where M denotes a parameter of the theory. The Starobinsky model is not plagued by the graceful exit problem and the period of cosmic acceleration is followed by the radiation dominated epoch with a transient matter dominated phase [Sta81, Vil85, MMS86]. Moreover it predicts nearly scale-invariant spectra of gravitational waves and temperature anisotropies consinstent with CMB observations [KMP87, HN01, DT10].

Although the Starobinsky model is the most studied among the $f(R)$ theories, there are also other inflationary models based on different choices of $f(R)$, for example, exponential [Zha06] or R^4 [KKW10a, KKW10b]. Some models even try to unify the late[3] and early time behaviours of the universe with a single gravitational action [Car04, NO07, Cog08]. For all these models we now study the spectra of scalar and tensor perturbations generated during inflation.

For $f(R)$ gravity the scalar field ϕ and all its contributions are absent. We can thus set $\phi = \delta\phi = 0$ in all the results derived in the previous section. For example, now we have $\epsilon_2 = 0$. A problem arises from the function E which happens to be ill-defined since divided by ϕ. The way out, in order to save our results, is to redefine it as $E := \frac{3}{2}\dot{F}^2$. In this manner we have [HN01, DT10]

$$Q_m = \frac{6F\epsilon_3^2}{(1+\epsilon_3)^2} = \frac{E}{FH^2(1+\epsilon_3)^2}\,, \tag{5.57}$$

and

$$\epsilon_4 = \frac{\ddot{F}}{H\dot{F}}\,. \tag{5.58}$$

The background equations (5.8) and (5.9) reduce to

$$3FH^2 = \frac{1}{2}(RF - f) - 3H\dot{F}\,; \tag{5.59}$$

$$-2F\dot{H} = \ddot{F} - H\dot{F}\,. \tag{5.60}$$

[3] In these models the late time acceleration results as an effect of the curvature rather than being driven by some kind of Dark Energy.

Eq. (5.60) can be rewritten in terms of the slow-roll parameters as

$$\epsilon_1 = -\epsilon_3(1-\epsilon_4)\,, \tag{5.61}$$

which, to first order, gives the following constraint

$$\epsilon_1 + \epsilon_3 \simeq 0\,. \tag{5.62}$$

Hence, assuming $\epsilon_i \ll 1$ (with $i = 1, 3, 4$), for $f(R)$ theories the scalar spectral index (5.43) becomes

$$\begin{aligned} n_s - 1 &\simeq -4\epsilon_1 + 2\epsilon_3 - 2\epsilon_4 \\ &\simeq -6\epsilon_1 - 2\epsilon_4\,, \end{aligned} \tag{5.63}$$

while the tensor spectral index (5.43), as well as the tensor-to-scalar ratio, vanish at first order:

$$n_t \simeq 0\,; \tag{5.64}$$

$$r \simeq 0\,. \tag{5.65}$$

We can compare (5.63) and (5.64) with the results (4.120) and (4.132) obtained within the standard model of inflation. We notice that in (5.63) ϵ_1 appears with a factor 6 instead of 4, whilst ϵ_4 plays the role of ϵ_2 in (4.120). This reflects the fact that now inflation is driven by some gravitational modification rather than by a scalar field ϕ. Tuning the parameter ϵ_4 with a consistent choice of $f(R)$, could still reproduce the experimental number (4.122). The scalar spectral index n_s can thus be used in order to constrain different $f(R)$ models, but cannot discriminate between $f(R)$ and standard inflation.

The tensor spectral index n_t presents instead another story. At first order in the slow-roll parameters, which is the only one we will hopefully measure soon, the $f(R)$ inflation has no tensor modes. This is at odds with the standard model of inflation for which a minimal quantity of tensor modes is expected. The result can thus be used to discriminate between $f(R)$ and standard inflation: if even a small amount of tensor modes are observed by future CMB experiments, $f(R)$ theories have to be discarded as possible models of inflation.

5.5 Example 2: Starobinsky Inflation

In order to compute some numbers, we analyze the Starobinsky model (5.56) as a specific $f(R)$ inflationary model. We will constrain the only free parameter of the theory, namely M, imposing the few experimental data we have. For this section we do not set the Planck mass m_{pl} equal to one[4].

[4] The Planck mass, in our units, is the inverse of the Newton constant G.

For the Starobinsky model inflation occurs in the regime $R \gg M^2$ and $|\dot{H}| \ll H^2$. We can thus approximate F with[5]

$$F \simeq \frac{R}{3M^2} \simeq \frac{4H^2}{M^2}\,, \tag{5.66}$$

From the estimations (5.44) and (5.54), the scalar and tensor power spectra becomes, repectively,

$$\mathcal{P}_s \simeq \frac{1}{12\pi}\left(\frac{M}{m_{pl}}\right)^2 \frac{1}{\epsilon_1^2}\,; \tag{5.67}$$

$$\mathcal{P}_t \simeq \frac{4}{\pi}\left(\frac{M}{m_{pl}}\right)^2 ; \tag{5.68}$$

where we have used the constraint (5.62). Without going too much into details, for Starobinsky inflation the number of e-foldings N happens to be related to ϵ_1 as [HN01, DT10]

$$N = \int_{t_i}^{t_f} H\,dt \simeq \frac{1}{2\epsilon_1}\,. \tag{5.69}$$

The scalar power spectrum (5.67) becomes then

$$\mathcal{P}_s \simeq \frac{N^2}{3\pi}\left(\frac{M}{m_{pl}}\right)^2 . \tag{5.70}$$

Imposing the WMAP 5-year normalization [WMAP09] corresponding to

$$\mathcal{P}_s = (2.445 \pm 0.096) \times 10^{-9}\,, \tag{5.71}$$

and taking the tipical value $N \simeq 62$ (see (4.29)), we obtain the following number for the parameter M:

$$M \simeq 2.7 \times 10^{-6}\, m_{pl}\,. \tag{5.72}$$

We can also evaluate the scalar spectral index for this model. From the relation (5.66) we gain another constraint on the slow-roll parameters:

$$\epsilon_1 + \epsilon_4 \simeq 0\,. \tag{5.73}$$

Then, the result (5.63) reduces to

$$n_s - 1 \simeq -4\epsilon_1 \simeq -\frac{2}{N}\,, \tag{5.74}$$

[5] The last step is achieved substituting $R = 6(2H^2 + \dot{H})$, which comes from evaluating the trace of the field equation in the FRW metric.

which, for $N \simeq 62$, gives

$$n_s \simeq 0.968\,, \tag{5.75}$$

in agreement with the observed result (4.122). Finally, we report the (second-order) result for the tensor-to-scalar ratio [HN01, DT10]

$$r \simeq \frac{12}{N^2} \simeq 3.1 \times 10^{-3}\,. \tag{5.76}$$

We conclude discussing the results obtained. From (5.76) we immediately realize that the constraint $r < 0.22$ is satisfied. However, this result completely differs from the canonical one (4.135) being at second order in the slow-roll parameters. In fact, the value (5.76) is a hundred times smaller than the one expected within the standard model of inflation, which is of order 10^{-1}. The result (5.75) instead tells us that, with the value (5.72) for M, the Starobinsky model matches the experimental data at least for scalar perturbations. Note that the value (5.72) implies small deviations from canonical GR as long as $R \ll M^2$, i.e. in the low-energy limit.

The Starobinsky model can be considered the most viable alternative to the standard model of inflation. Once the future experiments will permit to measure the scalar-to-tensor ratio r with the required accuracy, we will know which one among the scalar field and gravitational modifications is the final answer to the inflationary riddle.

5.6 Example 3: Scalar-Tensor Theories

In this final section we focus on Scalar-Tensor theories studying how the general results derived in section 5.3 reduce. Scalar-Tensor inflationary models abound in literature and providing a complete bibliography is impossible in this work. However, we give some indicative references for the reader who wants to deepen the argument.

The most studied inflationary models within Scalar-Tensor gravity are (of course) based on Brans-Dicke theory. These models gave even rise to a new kind of inflation called extended inflation [MJ84, LS89, BM90, GQ90]. A generalization, where ω is considered an arbitrary function of ϕ, has been named hyperextended inflation [SA90]. These has been, for long time, the most studied models of Scalar-Tensor inflation, as denoted by the huge amount of technical papers studying the topic.

Another interesting model is the non-minimal inflation [FU90, NS10, ORS10], where the inflaton ϕ is non-minimally coupled to the curvature scalar R through the function $F(\phi) = 1+\xi\phi^2$. This model has been recently used as way to identificate the inflaton with the Standard Model Higgs boson [BS08, BKS08], where, anyway, some serious problems seem to arise[6].

[6]Non-minimally coupled inflation will be treated, within the Palatini formulation, in section 6.6.

Furthermore, there are many other example of inflation in Scalar-Tensor gravity [Pal10] and even models where the Brans-Dicke field does not act as the inflaton and two scalar field are introduced [Bar95, GW95].

To conclude we cite some works where the theory of cosmological perturbations has been employed in Scalar-Tensor inflation [SBB89, Kai95] (see also [Far04]). The following analysis includes all the cited models, the ones with two scalar fields.

In Scalar-Tensor theories we set $f(R,\phi) = F(\phi)R$ leaving $\omega(\phi)$ and $V(\phi)$ arbitrary. The background equations (5.8) and (5.9) reduce to

$$3FH^2 = -3H\dot{F} + \frac{1}{2}\omega\dot{\phi}^2 + V\,; \tag{5.77}$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + \omega\dot{\phi}^2\,; \tag{5.78}$$

where the function $F := \partial f/\partial R$ depends now only by the scalar field ϕ. Using the relation $\dot{F} = F_{,\phi}\dot{\phi}$, we can rewrite eq. (5.78) as

$$\epsilon_1 + \epsilon_3 = \epsilon_3\left(\epsilon_1 + 2\epsilon_3\right) + \epsilon_3^2\,\frac{2\omega F}{F_{,\phi}^2}\,. \tag{5.79}$$

Once given the functions F and ω, the factor $2\omega F/F_{,\phi}^2$ is simply a function of ϕ where no further ϵ's contribution appears[7]. Then, at first order, eq. (5.79) leads to the constraint

$$\epsilon_1 + \epsilon_3 \simeq 0\,, \tag{5.80}$$

which is nothing but the same constraint we have obtained in $f(R)$ gravity, eq. (5.62). The scalar spectral index (5.43) becomes

$$n_s - 1 \simeq -6\epsilon_1 - 2\epsilon_2 - 2\epsilon_4\,. \tag{5.81}$$

This result is similar to the one obtained in $f(R)$ gravity, but now the parameter ϵ_2 does not vanish. Because of (5.80), the tensor spectral index (5.53) and the tensor-to-scalar ratio (5.55) are negligible to first order,

$$n_t \simeq 0\,; \tag{5.82}$$

$$r \simeq 0\,; \tag{5.83}$$

exactly as it happens in $f(R)$ gravity.

We finally discuss these results. The scalar spectral index (5.81), a part from the factor 6 in front of ϵ_1, presents a deviation from the canonical result (4.120) depending on the new parameter ϵ_4. The presence of ϵ_2 still indicates that inflation is driven by the scalar field ϕ, but the ϵ_4 term reflects the non-trivial coupling between ϕ and the curvature scalar R. As in $f(R)$ gravity,

[7] For example, in Brans-Dicke theories, where $F = \phi$ and $\omega = \omega_0/\phi$, it becomes constant: $2\omega_0$.

an intelligent choice of the coupling function $F(\phi)$ could still reproduce the experimental number (4.122). The result (5.81) can then be used in order to constrain different Scalar-Tensor models of inflation, but cannot act as a tool to distinguish those models from the standard one.

The tensor spectral index (5.82), or equivalently the tensor-to-scalar ratio (5.83), presents instead a crucial difference from the canonical result (4.132). Similarly to the $f(R)$ case (5.64), it vanishes at first order in the slow-roll parameters. Hence, if only a small amount of primordial gravitational waves will be measured by forthcoming experiments, all the Scalar-Tensor models of inflation will be ruled out[8]. As a general conclusion, we can state that, within the metric formulation, any $f(R)$ or Scalar-Tensor modification of inflationary gravity provides a physically detectable mark in the tensor spectral index.

[8]Note that, though it can be considered as a particular Scalar-Tensor theory with $F = \omega = 1$, the results (5.81) and (5.83) do not hold for the standard model of inflation. This is due to the fact that the constraint (5.80) cannot be true anymore, since in eq. (5.79) we are now dividing by zero.

Chapter 6

Inflationary Perturbations in Modified Gravity: Palatini Formulation

In this chapter we repeat all the inflationary perturbation analysis we carried out in the previous chapter. Instead of using the metric formulation of Generalized Gravity, we will consider the Palatini approach to the theory. Again, the aim is to derive and discuss scalar and tensor spectral indices and to show how they reduce in both $f(R)$ and Scalar-Tensor theories.

The procedure mirrors the one of chapter 5. We first introduce the background equations and present the general treatment of cosmological perturbations for Palatini Generalized gravity, deriving in particular the spectral indices and the tensor-to-scalar ratio. Then we will study the different cases of $f(R)$ and Scalar-Tensor gravity, providing as specific example with non-minimally coupled gravity, which can be included in the Scalar-Tensor theories class.

Also for this chapter, the style of the text is kept at a high technical level, so several derivations are not written down explicitly but references to the original works are always provided.

The subject of cosmological perturbations in Palatini modified gravity has been studied in [KK06, LC06, ULT07, TUT08, DT10], whilst the applications to the inflationary universe have been analyzed by the author in [TC10].

6.1 Background Cosmological Equations

In this section we show how the background (cosmological) equations (4.8) and (4.9) are modified within the Palatini formulation of Generalized Gravity introduced in section 3.5. For the sake of simplicity, we recall the action

(3.72),

$$S_{GG} = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(\hat{R},\phi) - \frac{1}{2}\omega(\phi)(\partial\phi)^2 - V(\phi) + \mathcal{L}_M(g_{\mu\nu},\Psi)\right]\,, \quad (6.1)$$

and the field equations (3.73) and (3.75),

$$F(\hat{R},\phi)\hat{R}_{\mu\nu} - \frac{1}{2}f(\hat{R},\phi)g_{\mu\nu} = T_{\mu\nu} + T^{(\phi)}_{\mu\nu}\,; \quad (6.2)$$

$$\hat{\nabla}^\mu\hat{\nabla}_\mu\phi + \frac{1}{2\omega}\left[\omega_{,\phi}\partial^\mu\phi\partial_\mu\phi - 2V_{,\phi} + f(\hat{R},\phi)_{,\phi}\right] = 0\,; \quad (6.3)$$

obtained varying (6.1) independently with respect to $g^{\mu\nu}$ and ϕ. All the required notation has already been introduced in chapter 3, we just notice and stress that in the Palatini formulation the curvature invariant appearing in action (6.1) is not the usual curvature invariant defined by the Levi-Civita connection $\Gamma^\lambda_{\mu\nu}$ but is built with the independent connection $\hat{\Gamma}^\lambda_{\mu\nu}$. As we saw in section 3.5, an independent variation of action (6.1) with respect to $\hat{\Gamma}^\lambda_{\mu\nu}$ leads to the condition

$$\hat{\nabla}_\mu\left[\sqrt{-g}\,F(\hat{R},\phi)\,g^{\alpha\beta}\right] = 0\,, \quad (6.4)$$

which implies the following relation between the two connections:

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2F}\left[2\delta^\lambda_{(\mu}\partial_{\nu)}F + g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma F\right]\,. \quad (6.5)$$

Finally, we recall that using the condition (6.5) we can relate the two Ricci tensor as

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2F^2}(\nabla_\mu F)(\nabla_\nu F) - \frac{1}{F}\nabla_\mu\nabla_\nu F - \frac{1}{2F}g_{\mu\nu}\nabla_\sigma\nabla^\sigma F\,. \quad (6.6)$$

Having reassumed the main results of section 3.5 in the Palatini formulation, we can now focus on the cosmological applications. Once again, we assume the (background) universe isotropic and homogeneous and thus still described by the FRW metric[1] (4.1). The only non vanishing components of (6.6) are then [MW03, MW04, WM04]

$$\begin{aligned}\hat{R}_{00} &= -3\frac{\ddot{a}}{a} + \frac{3\dot{F}^2}{2F^2} - \frac{3}{2F}\nabla_0\nabla_0 F\,; && (6.7)\\ \hat{R}_{ij} &= [a\ddot{a} + 2\dot{a}^2 + 2K + \frac{a^2}{2F}\nabla_0\nabla_0 F]\delta_{ij} + \frac{\dot{F}}{F}\Gamma^0_{ij}\,; && (6.8)\end{aligned}$$

which, substituted back in eq. (6.2), lead to

$$\begin{aligned}6F\left(H^2 + \frac{\dot{F}^2}{4F^2} + \frac{H\dot{F}}{2F} + \frac{K}{a^2}\right) - f &= 2\omega\dot{\phi}^2 - 2V + \rho_M + 3P_M\,; && (6.9)\\ 2F\left(\dot{H} + \frac{K}{a^2}\right) - \frac{3\dot{F}^2}{2F} - H\dot{F} + \ddot{F} &= -\omega\dot{\phi}^2 - \rho_M - P_M\,; && (6.10)\end{aligned}$$

[1] See footnote at page 68.

where we used the definitions (5.6) and (5.7) for the homogeneous scalar field ϕ. These correspond to the modification of the canonical equations (4.8) and (4.10): the first one governs the evolution of the scale factor in the cosmology of Palatini Generalized Gravity, the second one will turn out useful later on. We remind that the functions f and F appearing in (6.9) and (6.10) are still functions of $\hat{R}$, but can in principle become function of R taking the trace of the field equations[2].

The background equations (6.9) and (6.10) differ not only form the canonical ones (4.8) and (4.10), but also from the corresponding ones obtained within the metric formulation (5.8) and (5.9). Of course this follows from the fact that the metric and Palatini formulations of Generalized Gravity correspond to two different physical theories. In any case, setting $f = \hat{R}$, $\omega = 1$ and $K = \rho_M = P_M = 0$, eqs. (6.9) and (6.10) still reduce to (4.8) and (4.10) obtained in standard GR.

Finally, though not of interest in what follows, using the FRW metric the modified KG equation (6.3), becomes

$$\ddot{\phi} + 3H\dot{\phi} + \frac{3\dot{F}}{F}\dot{\phi} + \frac{1}{2\omega}\left(\omega_{,\phi}\dot{\phi}^2 + 2V_{,\phi} - f_{,\phi}\right) = 0\,. \tag{6.11}$$

Note the differences with the metric equation (5.10). A part from the fact that the functions f and F depend now by $\hat{R}$ rather than by R, eq. (6.11) presents a new term in $\dot{\phi}$ which was absent in (5.10). This term comes from expanding the d'Alembertian operator $\hat{\nabla}^\mu\hat{\nabla}_\mu$ in eq. (6.3), which is now defined with the indepedent connection $\hat{\Gamma}^\lambda_{\mu\nu}$ and not, as usual, with the Levi-Civita connection $\Gamma^\lambda_{\mu\nu}$.

6.2 Cosmological Perturbations

In this section we develop the general formalism of cosmological perturbations presenting equations for scalar, vector and tensor perturbations in the Palatini formulation. Once again, all the material discussed in section 4.4 can be equally applied to Palatini Generalized Gravity, except for the perturbation equations which differ not only from the canonical ones, but also from the ones obtained within the metric formulation. This happens because the cosmological background equations are different in the metric and Palatini approaches.

Exactly as before, cosmological perturbations are divided in scalar (α, β, γ, ψ), vector (b_i, c_i) and tensor (h_{ij}) perturbations according to their transformation properties under spatial rotations. We still have the gauge

[2]See section 3.5.

invariants $\Phi_{(\alpha)}$, $\Phi_{(\psi)}$, $\Phi_i^{(V)}$, $\mathcal{R}_{\delta\phi}$ and[3]

$$\mathcal{R}_{\delta F} := \psi - \frac{H}{\dot{F}}\delta F\,, \tag{6.12}$$

while the quantities (4.83) will help to simplify the equations.

Scalar Perturbations. Perturbing the $G^0{}_0$, $G^0{}_i$, $G^i{}_j - \frac{1}{3}\delta^i_j G^k{}_k$ and $G^k{}_k - G^0{}_0$ components of the field equations leads to the following scalar perturbation equations [KK06, TUT08, TC10]

$$\begin{aligned} ZA + \frac{3K+\triangle}{a^2}\psi + \frac{1}{2F}\left[\frac{3\dot{F}^2}{2F} + 3H\dot{F} - \omega\dot{\phi}^2\right]\alpha \\ = \frac{1}{2F}\Bigg\{-\frac{1}{2}\left[\omega_{,\phi}\dot{\phi}^2 + (2V-f)_{,\phi}\right]\delta\phi \\ + \left[3H^2 - \frac{3\dot{F}^2}{4F^2} - \frac{R}{2} + \frac{3K^2-\triangle}{a^2}\right]\delta F \\ -\omega\dot{\phi}\dot{\delta\phi} + 3Z\dot{\delta F} + \delta\rho_M\Bigg\}\,; \end{aligned} \tag{6.13}$$

$$Z\alpha - \dot{\psi} + \frac{K}{a^2}\chi = \frac{1}{2F}\left[\omega\dot{\phi}\delta\phi - \left(H + \frac{3\dot{F}}{2F}\right)\delta F + \dot{\delta F} - \delta q_M\right]\,; \tag{6.14}$$

$$\dot{\chi} + \left(H + \frac{\dot{F}}{F}\right)\chi - \alpha - \psi = \frac{1}{F}\delta F\,; \tag{6.15}$$

$$\begin{aligned} \dot{A} + (H+Z)A + \frac{3\dot{F}}{2F}\dot{\alpha} + \left[6\dot{Z} - 3\dot{H} + \frac{3H\dot{F}}{2F} + \frac{2}{F}\omega\dot{\phi}^2 + \frac{\triangle}{a^2}\right]\alpha \\ = \frac{1}{2F}\Bigg\{4\omega\dot{\phi}\dot{\delta\phi} + \left[2\omega_{,\phi}\dot{\phi}^2 + (f-2V)_{,\phi}\right]\delta\phi \\ + \left[\frac{3\dot{F}^2}{F^2} + 6(H^2+\dot{H}) - R - \frac{\triangle}{a^2}\right]\delta F \\ + \left(3H - \frac{6\dot{F}}{F}\right)\dot{\delta F} + 3\ddot{\delta F} + \delta\rho_M + \delta P_M\Bigg\}\,; \end{aligned} \tag{6.16}$$

where there are no anisotropic stresses ($\Pi_{\mu\nu} = 0$) and the function Z is defined by

$$Z := H + \frac{\dot{F}}{2F}\,. \tag{6.17}$$

As it happens in the metric approach, eqs. (6.13)-(6.16) appear rather more complicated than the canonical ones (4.84)-(4.87). This is due to several

[3]See definition (5.13).

new terms depending now by the perturbation of the modifying function δF. Anyway, with an intelligent choice of gauge, we will be able again to recast these equations into (easy) handle relations.

Vector and Tensor Perturbations. The equations for vector and tensor perturbations in Palatini formalism are the same of the corresponding ones in the metric formulation. This is due to the fact that both R and ϕ do not have either vector or tensor components being pure scalars [KK06]. We can thus simply recall the results of section 5.2.

The vector perturbation equation is

$$\frac{\triangle + 2K}{2a^2}\Phi_i^{(V)} = \frac{\delta q_i^M}{F}\,, \tag{6.18}$$

whose right hand side vanishes in case of no matter fields. Once again, eq. (6.18) tells us that vector perturbations will not be generated during inflation, making these modes uninteresting in Palatini Generalized Gravity too. The gravitational wave (amplitude) equation, the equation for tensor perturbations, is given by

$$\ddot{h} + \left(3H + \frac{\dot{F}}{F}\right)\dot{h} + \frac{2K - \triangle}{a^2}h = 0\,. \tag{6.19}$$

Although the form is the same, eq. (6.19) is slightly different from eq. (5.21) since F is now function of $\hat{R}$ rather than of R.

6.3 Perturbations from Inflation

This section is dedicated to present the original work the author carried out in [TC10]. We will study cosmological perturbations generated during the primordial inflationary phase within the Palatini Generalized Gravity framework. All the obtained results will be completely general, leaving the specific cases for the last sections.

Once again, we need to choose a convenient gauge first. The issue is identical to the one discussed in section 5.3, so we limit us to reassume only the main points here. A small difference appears in the fact that now the function f depends on $\hat{R}$ instead of R, but all the other arguments remain the same.

We will select either the Uniform-field gauge $\delta\phi = 0$ or the Uniform-F gauge $\delta F = 0$ depending on which theory we are working with. The Uniform-field gauge will be employed in $f(R)$ theories where the scalar field ϕ is absent, while the Uniform-F gauge will be useful in Scalar-Tensor gravity where we have $f(\hat{R},\phi) = F(\phi)\hat{R}$. In both cases, the choice we make leads to the condition $\delta\phi = \delta F = 0$, which simplifies enormously the scalar perturbation equations (6.13)-(6.16) and renders ψ gauge invariant. In fact,

ψ can be replaced by $\mathcal{R}$ wherever it appears in our equations since we get $\mathcal{R} = \mathcal{R}_{\delta F} = \mathcal{R}_{\delta\phi} = \psi$.

Finally, we remind that the following analysis does not hold for more general theories where both a scalar field and non-linear gravitational modifications are considered.

Scalar Perturbations

In what follows we focus on inflation setting as usual $K = 0$ (spatially flat spacetime) and $T_{\mu\nu} = 0$ (no matter fields). Moreover, we impose our gauge $\delta\phi = \delta F = 0$, substitute ψ with $\mathcal{R}$ and work in (spatial) Fourier space ($\triangle = -k^2$). With these assumptions all the right hand sides of eqautions (6.13)-(6.16) vanish and we have

$$ZA - \frac{k^2}{a^2}\mathcal{R} + \frac{1}{2F}\left[\frac{3\dot{F}^2}{2F} + 3H\dot{F} - \omega\dot{\phi}^2\right]\alpha = 0\,; \tag{6.20}$$

$$Z\alpha - \dot{\mathcal{R}} = 0\,; \tag{6.21}$$

$$\dot{\chi} + \left(H + \frac{\dot{F}}{F}\right)\chi - \alpha - \mathcal{R} = 0\,; \tag{6.22}$$

$$\dot{A} + (H + Z)\,A + \frac{3\dot{F}}{2F}\dot{\alpha} + \left[6\dot{Z} - 3\dot{H} + \frac{3H\dot{F}}{2F} + \frac{2}{F}\omega\dot{\phi}^2 - \frac{k^2}{a^2}\right]\alpha = 0\,. \tag{6.23}$$

Again, from eq. (6.21) we have

$$\alpha = \frac{\dot{\mathcal{R}}}{Z}\,, \tag{6.24}$$

which inserted into (6.20) gives

$$A = \frac{1}{Z}\left[\frac{k^2}{a^2}\mathcal{R} - \frac{\dot{\mathcal{R}}}{2FZ}\left(\frac{3\dot{F}^2}{2F} + 3H\dot{F} - \omega\dot{\phi}^2\right)\right]\,. \tag{6.25}$$

Putting eqs. (6.24) and (6.25) into eq. (6.23) and using the background equation (6.10), we obtain a second-order differential equation for the curvature invariant $\mathcal{R}$:

$$\ddot{\mathcal{R}} + \left(3H + \frac{\dot{Q}_P}{Q_P}\right)\dot{\mathcal{R}} + \frac{k^2}{a^2}\mathcal{R} = 0\,, \tag{6.26}$$

where we have defined

$$Q_P := \frac{\omega\dot{\phi}^2}{Z^2}\,. \tag{6.27}$$

The form of eq. (6.26) is equal to the one we found in standard inflation, eq. (4.105). Once again, the differences show up in the definition of the function Q_P, which not only differs from the canonical Q (4.106) but also from the metric Q_m (5.29). In the Palatini formulation of the theory, the term depending on the modifying function F appearing in Q_m, is completely absent. This is exactly the crucial point determining the differences between the Palatini and metric formulations in the physical results we are going to derive.

The analysis can now proceed in parallel to the ones we carried out in the previous chapters. Eq. (6.26) can be rewritten in a useful form by defining suitable Mukhanov-Sasaki variables $z_s := a\sqrt{Q_P}$ and $u_s := z_s\mathcal{R}$, which allows us to transform it into the equation for an oscillator with a time dependent frequency which is formed only by background quantities:

$$u_s'' + \left(k^2 - \frac{z_s''}{z_s}\right) u_s = 0\,. \tag{6.28}$$

Remind that a prime denotes differentiation with respect to the conformal time η defined in (4.2). In analogy with the standard and metric cases, we define the following slow-roll parameters

$$\epsilon_1 := -\frac{\dot{H}}{H^2}\,, \qquad \epsilon_2 := \frac{\ddot{\phi}}{H\dot{\phi}}\,, \qquad \epsilon_3 := \frac{\dot{F}}{2HF}\,. \tag{6.29}$$

The fourth parameter ϵ_4, appearing in the metric analysis, is usless in the Palatini case and will not be introduced. In terms of parameters (6.29) the quantity Q_P reads[4]

$$Q_P = \frac{\omega\dot{\phi}^2}{H^2(1+\epsilon_3)^2}\,. \tag{6.30}$$

Assuming again the inflationary conditions

$$\epsilon_i \ll 1 \quad \text{and} \quad \dot{\epsilon}_i = 0 \quad \text{with} \quad i = 1, 2, 3\,, \tag{6.31}$$

we have

$$\frac{z_s''}{z_s} = \frac{\nu_s^2 - 1/4}{\eta^2}\,, \tag{6.32}$$

where we have used relation (4.108) and

$$\nu_s^2 := \frac{1}{4} + \frac{(1+\epsilon_1+\epsilon_2)(2+\epsilon_2)}{(1-\epsilon_1)^2}\,. \tag{6.33}$$

[4]Note that if we want to write Q_P exactly as we wrote Q_m in (5.34), we just have to define the function $E := \omega F$, in which case we get $\epsilon_4 = \epsilon_3$. Then, all the results obtained within the metric approach can be equally applied to the Palatini formulation if we just take into account the condition $\epsilon_4 = \epsilon_3$.

Note that now, in the definition of ν_s, the parameter ϵ_3 does not appear anywhere[5]. The solution of eq. (6.28) is then

$$u_s = \frac{1}{2}\sqrt{\pi|\eta|}e^{i\pi(1+2\nu_s)/4}H^{(1)}_{\nu_s}(k|\eta|)\,, \tag{6.34}$$

which we have normalised by taking the early time, short wavelength ($k\eta \to \infty$) limit of the general solution, which reduces to the adiabatic vacuum case for a field in an expanding background $u_s \to e^{-ik\eta}/\sqrt{2k}$.

In the super-horizon limit ($k\eta \to 0$), the power spectrum of scalar perturbations, defined in (4.116), becomes

$$\mathcal{P}_s := \frac{k^3}{2\pi^2}|\psi|^2 \simeq \frac{1}{Q_P}\left((1-\epsilon_1)\frac{\Gamma(\nu_s)}{\Gamma(3/2)}\frac{H}{2\pi}\right)^2\left(\frac{|k\eta|}{2}\right)^{3-2\nu_s}\,. \tag{6.35}$$

Since the scalar perturbation is conserved after exiting the Hubble radius we can evaluate the power spectrum (6.35) at the time when $k = aH$ and examine its spectral index n_s, defined in (4.118), as

$$n_s - 1 := \left.\frac{d\ln\mathcal{P}_s}{d\ln k}\right|_{k=aH} = 3 - 2\nu_s\,. \tag{6.36}$$

During inflation we have $\epsilon_i \ll 1$ ($i = 1, 2, 3$), hence, at first order in the ϵ's, the scalar spectral index becomes

$$n_s - 1 \simeq -4\epsilon_1 - 2\epsilon_2\,. \tag{6.37}$$

Moreover, from eq. (6.35) follows an useful rapid estimation of the scalar power spectrum:

$$\mathcal{P}_s \simeq \frac{1}{Q_P}\left(\frac{H}{2\pi}\right)^2\,. \tag{6.38}$$

We finally compare these results with the corresponding ones found in the standard and metric cases. The scalar spectral index (6.37) is identical to the standard case arising from the Einstein-Hilbert action minimally coupled to an inflaton field (4.120). It is however distinct from the Generalized Gravity case in metric formalism (5.43) in that the scalar spectral index n_s has no explicit dependence, at linear order, on ϵ_3, or rather, on the choice of the function f. This means that, as long as the scalar field ϕ plays the same role as in the standard model of inflation, in Palatini Generalized Gravity the observable n_s, at first order in the slow-roll parameters, is physically indistinguishable from the canonical result (4.120). In particular we learn that, in the Palatini formulation, any modified theory of gravity will reproduce the experimental number (4.122), and thus, at least on the scalar perturbation side, will be physically equivalent to the standard inflationary model.

[5] Confront this definition with the standard one (4.110) and with the metric one (5.37) after the condition $\epsilon_3 = \epsilon_4$ has been deployed.

Tensor Perturbations

Gravitational waves in Palatini Generalized Gravity obey the following amplitude equation

$$\ddot{h} + \left(3H + \frac{\dot{F}}{F}\right)\dot{h} + \frac{k^2}{a^2}h = 0\,. \tag{6.39}$$

which is the same they obey in the metric formulation[6]. Consequently, the analysis of inflationary tensor perturbations is identical for both the approaches, so now we recall only the main results we found in section 5.3.

The tensor spectral index, at first order in the slow-roll parameters, is

$$n_t \simeq -2\epsilon_1 - 2\epsilon_3\,, \tag{6.40}$$

while the tensor-to-scalar ratio is

$$r := \frac{\mathcal{P}_s}{\mathcal{P}_t} \simeq \frac{8Q_m}{F} \simeq 16\,(\epsilon_1 + \epsilon_3)\,. \tag{6.41}$$

Of course, also in the Palatini formulation the consistency relation (4.136) is satisfied: $r = -8n_t$. The tensor power spectrum can be estimated as

$$\mathcal{P}_t \simeq \frac{8}{F}\left(\frac{H}{2\pi}\right)^2\,. \tag{6.42}$$

Because of this equivalence between Palatini and metric tensor perturbations, all the results we are going to find in the next sections will be, at least at first order, similar to the ones we derived in chapter 5.

6.4 Example 1: $f(R)$ Theories

We first look at $f(R)$ theories where the acceleration of the inflationary expansion is driven solely by the $f(R)$ modifications to the action. Several indications suggest that Palatini $f(R)$ gravity is not a viable model of inflation because of many troubles in matching observations. As we now see doing perturbation analysis, the theory completely miss our physical universe.

Palatini $f(R)$ inflationary models have been studied, for example, for R^2 gravity [MW03, WM04], but the more promising attempts try to unify both the early and late time accelerations introducing positive as well as negative power corrections [Sot06a, Sot06b, FTT07]. Their (late time) perturbations has been considered in [LCC07, DT10].

More recently, some works employ the Palatini formulation of $f(R)$ gravity to reproduce an early time bouncing cosmology in order to avoid the Big

[6] See previous section.

Bang singularity[7] [BOS09, BOS10, Olm10, BO10]. Perturbation analysis for these models has been carried out in [Koi10].

The following result for inflationary perturbations in Palatini $f(R)$ gravity has been derived in [TC10].

In attempting to analyze inflationary cosmological perturbations in Palatini $f(R)$ gravity, an important constraint arises: whereas $f(R)$ theories in the metric formalism introduce an extra dynamical scalar degree of freedom, the same theory in the Palatini approach introduces only a non-dynamical scalar mode with an algebraic constraint. This is most clearly seen by relating both cases to Brans-Dicke theories with an explicit scalar degree of freedom exactly as we did in chapter 3. In the Palatini case the second-order terms in the scalar cancel out exactly resulting in a non-dynamical scalar mode, although this is not obvious when first inspecting the action.

This result is evident in our treatment in that the second-order term in (6.26) is proportional to Q_P which is not defined in the absence of the scalar mode ϕ. Indeed in this case the equation reduces to a first-order constraint

$$\left(3H+\frac{3\dot{F}}{2F}\right)\dot{\mathcal{R}}+\frac{k^2}{a^2}\mathcal{R}=0\,. \tag{6.43}$$

This shows that in the Palatini formulation of $f(R)$ theories any initial curvature perturbation decays during inflation and are not useful for seeding the growth of structures[8]. Thus, Palatini inflation driven solely by $f(R)$ modifications cannot be considered as a viable model since it does not reproduce the universe we live in. Physically, the roots of this lackness lie on the fact that Palatini $f(R)$ gravity does not introduce any degree of freedom more than GR; mathematically, this happens since Q_P in (6.27) does not present the modifying term appearing in Q_m (5.29).

Finally, though usless because of the considerations above, we look at tensor perturbations. In Palatini $f(R)$ gravity the background equation (6.10) reduces to

$$2F\dot{H}-\frac{3\dot{F}^2}{2F}-H\dot{F}+\ddot{F}=0\,, \tag{6.44}$$

which, in terms of the slow-roll parameters, can be rewritten as

$$(\epsilon_1+\epsilon_3)(1+\epsilon_3)=0\,. \tag{6.45}$$

At first order we get the constraint

$$\epsilon_1+\epsilon_3\simeq 0\,, \tag{6.46}$$

[7] Without violationg any energy conditions.

[8] Note that in eq. (6.43) the function F still depends on $\hat{R}$. However, if we replace $\hat{R}$ in favour of R with the aid of the trace of the field equations, no more gauge invariants come out. Thus, eq. (6.43) will still be a first-order differential equation for $\mathcal{R}$ and all the following considerations will remain.

which sets to zero the tensor spectral index (6.40) and the tensor-to-scalar ratio (6.41),

$$n_s \simeq 0 \qquad \text{and} \qquad r \simeq 0\,. \tag{6.47}$$

This result is analogous to the one we found in metric $f(R)$ gravity[9], meaning that, on the tensor perturbation side, the two formulations act identically.

6.5 Example 2: Scalar-Tensor Theories

The second example we propose is Scalar-Tensor gravity, which was introduced in section 3.1. The Palatini formulation of Scalar-Tensor theories has not been sistematically studied throughout the literature, so only sporadic works can be found [Lin76, BM97, Igl07]. The most interesting model is non-minimal inflation [BD08] which will be treated in the next section. For the moment, we analyze Scalar-Tensor theories in general, showing how the results derived in section 6.3 reduce.

In Palatini Scalar-Tensor gravity we have $f(\hat{R},\phi) = F(\phi)\hat{R}$, while $\omega(\phi)$ and $V(\phi)$ are arbitrary functions of ϕ. The background equations (6.9) and (6.10) becomes

$$6F\left(H^2 + \frac{\dot{F}^2}{4F^2} + \frac{H\dot{F}}{2F}\right) - F\hat{R} \;=\; 2\omega\dot{\phi}^2 - 2V\,; \tag{6.48}$$

$$2F\dot{H} - \frac{3\dot{F}^2}{2F} - H\dot{F} + \ddot{F} \;=\; -\omega\dot{\phi}^2\,; \tag{6.49}$$

where now the function F depends only by the scalar field ϕ. Using the relation $\dot{F} = F_{,\phi}\dot{\phi}$, eq. (6.49) can be rewritten in terms of the slow-roll parameters (6.29) as [TC10]

$$(\epsilon_1 + \epsilon_3)(1 + \epsilon_3) = \epsilon_3^2 \frac{2\omega F}{F_{,\phi}^2}\,. \tag{6.50}$$

At first order in the ϵ's this leads to the following constraint

$$\epsilon_1 + \epsilon_3 \simeq 0\,, \tag{6.51}$$

which is nothing but the condition (5.80) we obtained in the metric case[10]. The constraint (6.51) does not affect the scalar spectral index (6.37) since it does not present any ϵ_3 deviation

$$n_s - 1 \simeq -4\epsilon_1 - 2\epsilon_2\,. \tag{6.52}$$

[9] Note however the difference between eqs. (6.45) and (5.61), which implies that at second order the results are not the same.

[10] Although at first order this constraint is the same, note that at second order eq. (6.50) differs from eq. (5.79). This means that the following coincidence between the metric and Palatini results breaks up at the next order of approximation.

The tensor spectral index (6.40) and tensor-to-scalar ratio (6.41) are instead, once again, constrined to zero

$$n_s \simeq 0 \qquad \text{and} \qquad r \simeq 0\,. \tag{6.53}$$

In conclusion, although the Palatini formulation of scalar-tensor theories could drive inflation and produce nearly scale invariant scalar perturbations, in agreement with observations, they will not produce any tensor modes. This is a general features of Palatini Scalar-Tensor theories, which consequently might have to be discarded as possible models for inflation if even a small amount of tensor modes are observed by future CMB experiments.

6.6 Example 3: Non-Minimal Inflation

As an example of Scalar-Tensor gravity we analyse the case of non-minimally coupled inflation [FU90, NS10, ORS10] which has been recently used as an attempt to identify the inflaton field with the standard model Higgs boson [BS08, BKS08]. The following analysis is taken from [TC10].

In non-minimal inflation we have

$$F(\phi) = 1 + \xi\phi^2\,, \qquad \omega(\phi) = 1\,, \qquad V(\phi) = \lambda\left(\phi^2 - v^2\right)^2\,, \tag{6.54}$$

with ξ, λ, v parameters of the theory. The potential $V(\phi)$ should represent the Higgs symmetry-breaking potential. Slow-roll inflation is obtained in the region $\xi\phi^2 \gg v^2$, where the potential becomes $V(\phi) \simeq \lambda\phi^4$. Using the slow-roll conditions

$$\ddot{\phi} \ll H\dot{\phi}\,, \qquad \dot{\phi} \ll H\phi\,, \qquad \dot{\phi}^2 \ll V\,, \qquad \dot{H} \ll H^2\,, \tag{6.55}$$

from the background equations (6.9), (6.10) and (6.11) we derive the following first-order relations

$$H^2 \simeq \frac{\lambda\phi^4}{3\,(1+\xi\phi^2)}\,; \tag{6.56}$$

$$H\dot{\phi} \simeq \frac{4\lambda\phi^3}{3}\frac{\xi\phi^2 - 1}{\xi\phi^2 + 1}\,; \tag{6.57}$$

$$\dot{H} \simeq \frac{4\lambda\xi\phi^4}{3}\frac{\xi\phi^2 - 1}{(\xi\phi^2 + 1)^2}\,. \tag{6.58}$$

Then, from the definitions (6.29), the slow-roll parameters become

$$\epsilon_1 \simeq -4\xi\,\frac{\xi\phi^2 - 1}{\xi\phi^2 + 1}\,; \tag{6.59}$$

$$\epsilon_2 \simeq 4\xi \frac{(\xi\phi^2+3)(2\xi\phi^2-1)}{\xi\phi^2(\xi\phi^2+1)}\,; \tag{6.60}$$

$$\epsilon_3 \simeq 4\xi \frac{\xi\phi^2-1}{\xi\phi^2+1}\,. \tag{6.61}$$

According to the general constraint obtained above for Scalar-Tensor theories (6.51), we immediately note that ϵ_1 is exactly the opposite of ϵ_3. As expected then the tensor spectral index and tensor-to-scalar ratio vanish at first order. Using the relation (6.52) the scalar spectral index can be written as

$$n_s - 1 \simeq \frac{8}{\phi^2}\,\frac{3-7\xi\phi^2}{1+\xi\phi^2}\,. \tag{6.62}$$

In order to constrain ξ with this result we need a value for ϕ to substitute in. This can be obtained from the minimum number of e-foldings N which occurred during inflation,

$$\begin{aligned} N &= \int_{\phi_{\rm start}}^{\phi_{\rm end}} \frac{H}{\dot\phi}\,d\phi = \int_{\phi_{\rm start}}^{\phi_{\rm end}} \frac{H^2}{H\dot\phi}\,d\phi \\ &\simeq -\frac{1}{8\xi}\log(1-\xi\phi^2_{\rm start})\,, \end{aligned} \tag{6.63}$$

where $\phi_{\rm start}$ and $\phi_{\rm end}$ (with $\phi_{\rm start} \gg \phi_{\rm end}$) are the values of the inflaton field at the beginning and end of inflation respectively and we have used the relations (6.56) and (6.57) in the last step.

Considering[11] $N \simeq 62$ and an observed scalar spectral index of $n_s \simeq 0.96$, we obtain the following positive[12] value for ξ of

$$\xi \simeq 2.9\times 10^{-3}\,. \tag{6.64}$$

The value is small, which means small deviation from minimal coupling and agreement with the metric case result [NS10]. However, non-minimally coupled inflationary models in the regime $\xi \gg 1$ are also considered viable [BS08] (see also [BD08] for the Palatini approach). The latter are indeed the models which try to unify the inflaton field with the Higgs boson.

Finally, given the value for ξ and imposing the WMAP 5-year normalization for the curvature perturbation power spectrum [WMAP09]

$$\mathcal{P}_s \simeq \left.\left(\frac{H^2}{2\pi\dot\phi}\right)^2\right|_{aH=k} \simeq 2.4\times 10^{-9}\,, \tag{6.65}$$

we can also obtain a value for λ:

$$\lambda \simeq 2.5\times 10^{-14}\,. \tag{6.66}$$

[11] See (4.29) and (4.122).

[12] We could also obtain a negative value for ξ, but te corresponding value of λ would be negative implying a potential unbounded from below.

Again, we find an extremely small value for λ, which however seems to agree with the metric case [NS10].

To conclude, the obtained results of ξ and λ tell us that non-minimally coupled inflation in its Palatini formulation matches the few measured experimental data only for small deviations from minimal coupling and for nearly flat potentials. Although it might be of interest from the particle physics point of view, if only a small amount of primordial gravitational waves was detected by forthcoming observations, the model would be discarded because of its Scalar-Tensor nature.

Conclusion

We briefly summarize here the work presented in this thesis giving references to the chapters and sections where the topics were studied in details. The results for scalar and tensor spectral indices obtained throughout the text for different gravitational theories are compared and discussed.

After the (rather long) introduction (Chapter 1), where we provided an overview of modern cosmology, we studied the metric and Palatini variational principles (Chapter 2), which, for GR, are nothing but two different formulations of the same theory. We then moved to analyze some theoretical modifications of GR using both metric and Palatini formalisms (Chapter 3). The gravitational theories presented were: Scalar-Tensor gravity, $f(R)$ gravity and Generalized Gravity. For all these theories we found that the metric and Palatini approaches lead to physically different theories rather than, as in GR, to two different formulations of the same theory.

After this, we started to focus on cosmology providing first a short review on the standard model of inflation and the theory of cosmological perturbations (Chapter 4). There we explained how a primordial inflationary phase can solve some major cosmological problems and introduced the concept of slow-roll inflation. Furthermore, we presented the general set up of cosmological perturbations discussing the gauge issue and deriving evolution equations. We finally defined the scalar and spectral indices showing how they, in the standard model of inflation, can be written to first order in the slow-roll parameters.

The last two chapters are devoted to the main subject of this thesis. We applied the theory of cosmological perturbations to metric (Chapter 5) and Palatini (Chapter 6) Generalized Gravity. We focused on the inflationary phase by deriving how scalar and tensor spectral indices read in those theories. We made specific examples reducing the analysis to $f(R)$ and Scalar-Tensor gravity and provided some numbers to match with experimental data in two particular inflationary models: Starobinsky and non-minimal inflations.

We can now compare the main results we found for different gravitational theories. In the following table we summarize the scalar (n_s) and tensor (n_t) spectral indices for all the inflationary models introduced:

Theory	$n_s - 1$	n_t
Standard Inflation	$-4\epsilon_1 - 2\epsilon_2$	$-2\epsilon_1$
Metric $f(R)$ Gravity	$-6\epsilon_1 - 2\epsilon_4$	0
Metric Scalar-Tensor Gravity	$-6\epsilon_1 - 2\epsilon_2 - 2\epsilon_4$	0
Palatini $f(R)$ Gravity	-	0
Palatini Scalar-Tensor Gravity	$-4\epsilon_1 - 2\epsilon_2$	0

All the results are taken to first order in the slow-roll parameters (5.32). For any of these theories, a part Palatini $f(R)$ gravity, the consistency relation connecting the tensor spectral index to the tensor-to-scalar ratio holds: $r = -8n_t$. From the table above, it is easy to understand the theoretical differences between these models marked by the observables n_s and n_t.

Let first focus on the scalar perturbation side. One immediately notes that for Palatini $f(R)$ gravity the scalar spectral index does not appear. This is due to the fact that for this theory scalar perturbations cannot be generated during the inflationary phase (Section 6.4). We thus cannot even calculate n_s since any primordial gravitational wave would rapidly decay once inflation ended and consequently would not reach us today. In particular, all the late time structures of our universe would not shape since the early time scalar perturbations cannot act as seeds for their growth. For this reasons, the theory cannot be employed to build viable inflationary models. Theoretically, this happens because in Palatini $f(R)$ gravity we get the same dynamical degrees of freedom than GR, meaning there are no further scalar degrees of freedom to drive inflation. This last feature is clearer recasting the theory in the form of a Brans-Dicke theory where the further scalar degree of freedom should appear explicitly (Section 3.4).

Another great surprise comes from Palatini Scalar-Tensor gravity, where the result for n_s turns out to be identical to the canonical one (Section 6.5). This implies that, as long as the scalar field ϕ plays the same role of the inflaton, the theory cannot be physically distinguished from standard inflation measuring scalar perturbations only.

In the metric theories, the gravitational modification shows its effects through the slow-roll parameter ϵ_4 and, though of less importance, through the numerical factor in front of ϵ_1. In metric $f(R)$ gravity the slow-roll parameter ϵ_2 vanishes because the scalar field ϕ is clearly absent (Section 5.4). However, its role in the scalar spectral index is now played by ϵ_4, meaning that inflation is directly driven by the modified gravitational part of the action rather than by an external scalar field. In fact, contrary to the Palatini case, metric $f(R)$ gravity does present a scalar degree of freedom more than GR (Section 3.3). In metric Scalar-Tensor gravity inflation is instead driven by the scalar field ϕ as the appearence of ϵ_2 in n_s confirms (Section 5.6). Here the role of ϵ_4 is marginal and determines only small corrections to the almost standard result given by ϵ_1 and ϵ_2.

To conclude the discussion on scalar perturbations, we find different expressions for the scalar spectral index n_s depending on the gravitational theory considered. Anyway, all of them, except Palatini $f(R)$ gravity, are in some way similar, if not equal, each other. With a consinstent modification of the gravitational action all these scalar spectral indices can indeed reproduce the experimental number [WMAP09]

$$n_s = 0.960 \pm 0.013\,,$$

and thus be all empirically acceptable. This means that the scalar spectral index n_s cannot be used in order to distinguish between standard and modified inflation. It becomes however an important tool if one is interested to discriminate between different inflationary models within the same theoretical framework.

The tensor spectral index presents a completely different story. Looking at the table above, we realize that any minimal modification to the standard gravitational action, regardless of the variational principle in use, sets immediately the spectral index to zero. This first order result differs from the one we find in standard inflation where n_t is related to ϵ_1. Although it has not been measured yet, we can utilize an eventual observation of n_t in order to select which one between canonical and modified inflations really describe the early universe behaviour. If only a small amount of gravitational wave modes will be detected, any inflationary model based on a modified gravitational action would be experimentally ruled out. In numbers, the expected value of r, which is always related to n_s by $r = -8n_s$, is of order 10^{-1} for standard inflation while is at least one hundred times smaller for all the other models. At the moment we can only rely on an upper bound of $r < 0.22$ [WMAP09] and wait for future CMB experiments, which will hopefully permit us to know if modified gravitational theories represent a viable theoretical way to describe an inflationary primordial epoch.

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